Solution 1:

1. Moduli

The modulus of u is $|u| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

The modulus of v is $|v| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$

2. Arguments

The argument of u

$$u = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2}e^{i\frac{\pi}{4}}$$

The argument of u is $\pi/4$. (first quadrant).

The argument of v

$$v = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2e^{\frac{2i\pi}{3}}$$

The argument of u is $\pi/4$ (second quadrant).

3. Cube Roots of u

We seek the complex solutions of $z^3 = u \Rightarrow \sqrt{2}\left(cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = S^3\,(\cos 3\theta + i\sin 3\theta)$

The modulus of each cube root is $S^3 = \sqrt{2} = 2^{\frac{1}{2}} \Rightarrow S = 2^{\frac{1}{6}}$ or $S = \sqrt[6]{2}$

The arguments are $3arg(z) = 3\theta = \frac{\pi}{4} + 2\pi k, k \in \mathbb{Z} \Rightarrow \theta_k = \frac{\pi}{12} + \frac{2\pi}{3}k, k = 0,1,2$

$$\theta_k = \begin{cases} \dfrac{\pi}{12}, k = 0\ (first\ quadrant) \\ \dfrac{9\pi}{12} = \dfrac{3\pi}{4}, k = 1\ (Second\ quadrant) \\ \dfrac{17\pi}{12} = \pi + \dfrac{5\pi}{12}\ (Third\ quadrant)\ k = 2 \end{cases}$$

Hence the arguments of z (the cube roots of u) are: $\frac{\pi}{12}, \frac{3\pi}{4}$ and $\frac{-7\pi}{12}$ (in principal range)

4. Modulus and Argument of $\frac{u}{v}$

The modulus is $\left|\frac{u}{v}\right| = \frac{|u|}{|v|} = \frac{\sqrt{2}}{2}$

An argument

$$Arg(\frac{u}{v}) = arg(u) - arg(v) = \frac{\pi}{4} - \frac{2\pi}{3} = -\frac{5\pi}{12}$$

$$\frac{u}{v} = \frac{\sqrt{2}e^{i\frac{\pi}{4}}}{2e^{\frac{2i\pi}{3}}} = \frac{\sqrt{2}}{2}e^{i\left(\frac{\pi}{4}-\frac{2\pi}{3}\right)} = \frac{\sqrt{2}}{2}e^{-\frac{5i\pi}{12}} = \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{5\pi}{12}\right) + i\sin\left(-\frac{5\pi}{12}\right)\right)$$

5. Trigonometric Values

We have

$$\frac{u}{v} = \frac{1+i}{-1+i\sqrt{3}} = \frac{(1+i)(-1-i\sqrt{3})}{4} = \frac{-1+\sqrt{3}+i(-1-\sqrt{3})}{4}$$

Thus

$$\begin{cases} \frac{\sqrt{2}}{2}\cos\left(-\frac{5\pi}{12}\right) = \frac{-1+\sqrt{3}}{4} \\ \frac{\sqrt{2}}{2}\sin\left(-\frac{5\pi}{12}\right) = \frac{-1-\sqrt{3}}{4} \end{cases} \Leftrightarrow \begin{cases} \cos\left(-\frac{5\pi}{12}\right) = \frac{-1+\sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2}+\sqrt{6}}{4} \\ \sin\left(-\frac{5\pi}{12}\right) = \frac{-1-\sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2}-\sqrt{6}}{4} \end{cases}$$

Solution 2:

Let $u = \dfrac{w - \overline{w}z}{1-z}$

Since u is purely real, so

$u = \overline{u}$

$$\Rightarrow \left(\frac{w-\overline{w}z}{1-z}\right) = \left(\overline{\frac{w-\overline{w}z}{1-z}}\right)$$

$$= \left(\frac{\overline{w}-w\overline{z}}{1-\overline{z}}\right)$$

$\Rightarrow \quad (1-z)(w-\overline{w}z) = (1-z)(\overline{w}-w\overline{z})$

$\Rightarrow \quad w-\overline{w}z - wz + w\overline{w}z = \overline{w} - w\overline{z} - z\overline{w} + wz\overline{z}$

$\Rightarrow \quad (w-\overline{w})(z\cdot\overline{z}-1) = 0$

$\Rightarrow \quad (z\cdot\overline{z}-1) = 0$

$\Rightarrow \quad |z|^2 - 1 = 0$

$\Rightarrow \quad |z|^2 = 1$

$\Rightarrow \quad |z| = 1$

Thus, the set of values of $z = \{z : |z| = 1, z \neq 1\}$.

Solution 3:

A.

$\because A \times B = \{2, 3\} \times \{4, 5\}$

$= \{(2, 4), (2, 5), (3, 4), (3, 5)\}$

and $B \times C = \{4, 5\} \times \{5, 6\}$

$= \{(4, 5), (4, 6), (5, 5), (5, 6)\}$

$\therefore \ (A \times B) \cup (B \times C) = \{(2, 4), (2, 5), (3, 4), (3, 5),$

$(4, 5), (4, 6), (5, 5), (5\ 6)\}$

Now, $n\{(A \times B) \cup (B \times C)\} = 8$

B. We know that

$$\begin{aligned} & 0 \le x-[x] < 1 && \text{for all } x \in R \\ \Rightarrow\ & 1 \le 1+x-[x] < 2 && \text{for all } x \in R \\ \Rightarrow\ & 1 \le g(x) < 2 && \text{for all } x \in R \\ \Rightarrow\ & f(g(x)) = 1. && \text{for all } x \in R \end{aligned}$$

<u>Solution4:</u>

❖ $aRb \Leftrightarrow a - b$ is an even integer, $a, b \in I$

- $a - a = 0$ (even integer)

  ∴ (a, a) ∈ $\mathcal{R}$, ∀ a ∈ I

  ∴ R is reflexive relation.

- Let $(a, b) \in \mathcal{R} \Rightarrow$ (a − b) is an even integer.

  $\Rightarrow -(b - a)$ is an even integer.

  $\Rightarrow (b - a)$ is an even integer.

  $\Rightarrow (b, a) \in \mathcal{R}$

  ∴ R is symmetric relation.

- Now, let $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$

  Then, $(a - b)$ is an even integer and $(b - c)$ is an even integer.

  So, let

  $a - b = 2x_1, x_1 \in I$

  and

  $b - c = 2x_2, x_2 \in I$

  $\therefore (a - b) + (b - c) = 2(x_1 + x_2)$

  $\Rightarrow (a - c) = 2(x_1 + x_2) \Rightarrow a - c = 2x_3$

  $\therefore (a - c)\ is\ an\ even\ integer.$

  $\therefore a\mathcal{R}\,b\ and\ b\mathcal{R}\,c \Rightarrow a\mathcal{R}\,c$

  So, $\mathcal{R}$ is transitive relation.

- this relation is **not** antisymmetric.

- For antisymmetry, whenever $a\mathcal{R}b$ and $b\mathcal{R}a$, it must follow that $a = b$.
- Take $a = 2\ and\ b = 4$. Then $2 - 4 = -2$ is even, so $2\mathcal{R}4$; also $4 - 2 = 2$ is even, so $4\mathcal{R}2$, yet $2 \neq 4$. Thus the antisymmetry condition fails.

So the relation is reflexive, symmetric, and transitive (hence an equivalence relation), but not antisymmetric