Exam on Mathematics 2 - Solutions

Exercise 1: Indefinite Integrals

1.

$$\int (5x^4 - 3x^2 + 2)\,dx = \boxed{x^5 - x^3 + 2x + C}$$

Steps:

$$\int 5x^4\,dx = x^5,$$
$$\int -3x^2\,dx = -x^3,$$
$$\int 2\,dx = 2x.$$
$$\text{Sum:}\quad x^5 - x^3 + 2x + C$$

2.

$$\int \frac{3x^2+2}{x^3+2x}\,dx = \boxed{\ln|x^3+2x| + C}$$

Steps:

$$\text{Let } u = x^3 + 2x,$$
$$du = (3x^2+2)\,dx.$$
$$\text{Integral becomes:}\quad \int \frac{du}{u} = \ln|u| + C.$$

3.

$$\int e^{2x}\cos(3x)\,dx = \boxed{\frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x) + C}$$

Steps: Use integration by parts twice:

$$\text{Let } u = e^{2x}, \quad dv = \cos(3x)\,dx,$$
$$du = 2e^{2x}\,dx, \quad v = \frac{\sin 3x}{3}.$$

Repeat for the resulting integral and solve for the original.

Exercise 2: First-Order Differential Equations

1.

$$y = \boxed{\left(\frac{3x^2}{2} + C\right)^{1/3}}$$

Steps:

$$\frac{dy}{dx} = \frac{x}{y^2},$$
$$y^2\,dy = x\,dx,$$
$$\int y^2\,dy = \int x\,dx,$$
$$\frac{y^3}{3} = \frac{x^2}{2} + C.$$

2.

$$y = \boxed{x^2 + \frac{C}{x^2}}$$

Steps:

$$\begin{aligned} x\frac{dy}{dx} + 2y &= 4x^2, \\ \frac{dy}{dx} + \frac{2}{x}y &= 4x, \\ \text{Integrating factor:} \quad \mu(x) &= e^{\int \frac{2}{x}\,dx} = x^2, \\ \text{Multiply and integrate:} \quad y &= x^2 + \frac{C}{x^2}. \end{aligned}$$

Exercise 3: Matrices

1. **(a)**

$$A + B = \boxed{\begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}}$$

Steps:

$$\begin{bmatrix} 2+1 & 3+0 \\ -1+2 & 4+(-2) \end{bmatrix}$$

(b)

$$A \cdot B = \boxed{\begin{bmatrix} 8 & -6 \\ 7 & -8 \end{bmatrix}}, \quad B \cdot A = \boxed{\begin{bmatrix} 2 & 3 \\ 6 & -2 \end{bmatrix}}$$

Steps:

$$\begin{aligned} A \cdot B &= \begin{bmatrix} (2)(1) + (3)(2) & (2)(0) + (3)(-2) \\ (-1)(1) + (4)(2) & (-1)(0) + (4)(-2) \end{bmatrix}, \\ B \cdot A &= \begin{bmatrix} (1)(2) + (0)(-1) & (1)(3) + (0)(4) \\ (2)(2) + (-2)(-1) & (2)(3) + (-2)(4) \end{bmatrix}. \end{aligned}$$

(c)

$$B^T = \boxed{\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}}$$

Steps: Transpose rows and columns.

2.

$$\det(C) = \boxed{42}$$

Steps:

$$\begin{aligned} \det(C) &= 3 \cdot \det \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix} - (-1) \cdot \det \begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix} + 2 \cdot \det \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}, \\ &= 3(20) + 1(-2) + 2(-8) = 60 - 2 - 16 = 42. \end{aligned}$$

3.

$$X = \boxed{\begin{bmatrix} \frac{4}{21} \\ -\frac{8}{21} \\ \frac{11}{21} \end{bmatrix}}$$

Steps:

- Compute C^{-1} using $\det(C) = 42$ and adjugate matrix.
- Multiply $C^{-1}D$.

Total: 20 points