L'arbi Ben M'hidi University of Oum El Bouaghi
G.T.U Departement

L1 **Exam on Mathematics 1** **2024/2025**
Duration: 1h 30min

Instructions

- Answer all questions in the spaces provided.
- Show all your work. Partial credit will be awarded where appropriate.
- Calculators are not allowed, write clearly and legibly. Unreadable answers will not be graded.

Solutions

Exercise 1: Multiple Choice Questions

1. What is the derivative of $f(x) = \ln(x^2+1)$?
Solution: Using the chain rule:
$$f'(x) = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}.$$
Answer: $\boxed{\frac{2x}{x^2+1}}$.

2. Which of the following functions is continuous everywhere?
Solution:
   - $f(x) = \frac{1}{x}$ is discontinuous at $x = 0$.
   - $f(x) = \sqrt{x}$ is defined only for $x \geq 0$.
   - $f(x) = \tan(x)$ is discontinuous at $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.
   - $f(x) = e^x$ is continuous everywhere.

   Answer: $\boxed{f(x) = e^x}$.

3. What is the inverse function of $f(x) = 3x - 5$?
Solution: Solve $y = 3x - 5$ for x:
$$y = 3x - 5 \implies x = \frac{y+5}{3}.$$
Thus, $f^{-1}(x) = \frac{x+5}{3}$. **Answer**: $\boxed{f^{-1}(x) = \frac{x+5}{3}}$.

4. What is the derivative of $f(x) = \arcsin(x)$?
Solution: The derivative of $\arcsin(x)$ is:
$$f'(x) = \frac{1}{\sqrt{1-x^2}}.$$
Answer: $\boxed{\frac{1}{\sqrt{1-x^2}}}$.

5. What is the limit $\lim_{x\to 0} \frac{\sin(x)}{x}$?
Solution: This is a standard limit:
$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$
Answer: $\boxed{1}$.

Exercise 2: Derivative and Continuity Problems

1. Find the derivative of $f(x) = x^3 \cdot e^{2x}$.
   Solution: Use the product rule:
   $$f'(x) = \frac{d}{dx}(x^3) \cdot e^{2x} + x^3 \cdot \frac{d}{dx}(e^{2x}) = 3x^2 e^{2x} + x^3 \cdot 2e^{2x}.$$
   Simplify:
   $$f'(x) = e^{2x}(3x^2 + 2x^3).$$
   Answer: $\boxed{f'(x) = e^{2x}(3x^2 + 2x^3)}$.

2. Determine if the function $f(x) = \frac{x^2-4}{x-2}$ is continuous at $x = 2$. If not, classify the discontinuity.
   Solution: Simplify $f(x)$:
   $$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2 \quad (\text{for } x \neq 2).$$
   At $x = 2$, $f(x)$ is undefined. However, the limit exists:
   $$\lim_{x \to 2} f(x) = 4.$$
   Thus, $f(x)$ has a **removable discontinuity** at $x = 2$. **Answer**: $\boxed{\text{Removable discontinuity at } x = 2}$.

3. Compute the derivative of $f(x) = \arctan(x^2)$.
   Solution: Use the chain rule:
   $$f'(x) = \frac{1}{1 + (x^2)^2} \cdot 2x = \frac{2x}{1 + x^4}.$$
   Answer: $\boxed{f'(x) = \frac{2x}{1 + x^4}}$.

4. Evaluate $\lim_{x \to \pi/2} \frac{\cos(x)}{x - \pi/2}$.
   Solution: Substitute $h = x - \pi/2$. As $x \to \pi/2$, $h \to 0$:
   $$\lim_{h \to 0} \frac{\cos\left(\frac{\pi}{2} + h\right)}{h} = \lim_{h \to 0} \frac{-\sin(h)}{h} = -1.$$
   Answer: $\boxed{-1}$.

Exercise 3: Advanced Derivative and Limit Problems

1. Derive the function $f(x) = \ln(\sin(x)) + \cos^2(x)$.
   Solution: Differentiate term by term:
   $$f'(x) = \frac{1}{\sin(x)} \cdot \cos(x) + 2\cos(x) \cdot (-\sin(x)) = \cot(x) - 2\sin(x)\cos(x).$$
   Simplify:
   $$f'(x) = \cot(x) - \sin(2x).$$
   Answer: $\boxed{f'(x) = \cot(x) - \sin(2x)}$.

2. Find the derivative of $f(x) = \arcsin\left(\frac{x}{2}\right) + \sqrt{4 - x^2}$.
   Solution: Differentiate term by term:
   $$f'(x) = \frac{1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}} \cdot \frac{1}{2} + \frac{-x}{\sqrt{4 - x^2}}.$$
   Simplify:
   $$f'(x) = \frac{1}{\sqrt{4 - x^2}} - \frac{x}{\sqrt{4 - x^2}} = \frac{1 - x}{\sqrt{4 - x^2}}.$$
   Answer: $\boxed{f'(x) = \frac{1 - x}{\sqrt{4 - x^2}}}$.

3. Evaluate $\lim_{x\to 0} \frac{\tan(3x)}{\sin(5x)}$.
Solution: Use the small-angle approximations $\tan(3x) \approx 3x$ and $\sin(5x) \approx 5x$:

$$\lim_{x\to 0} \frac{\tan(3x)}{\sin(5x)} = \lim_{x\to 0} \frac{3x}{5x} = \frac{3}{5}.$$

Answer: $\boxed{\frac{3}{5}}$.

4. Consider the function

$$h(x) = \begin{cases} x^2 + 2x & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$$

(a) Show that $h(x)$ is continuous at $x = 0$.
Solution: Check the left-hand limit ($x \to 0^-$):

$$\lim_{x\to 0^-} h(x) = 0^2 + 2(0) = 0.$$

Check the right-hand limit ($x \to 0^+$):

$$\lim_{x\to 0^+} h(x) = \sqrt{0} = 0.$$

Since both limits equal $h(0) = 0$, $h(x)$ is continuous at $x = 0$. **Answer**: $\boxed{\text{Continuous at } x = 0}$.

(b) Find the derivative $h'(x)$ for $x < 0$ and $x > 0$.
Solution: For $x < 0$:

$$h'(x) = \frac{d}{dx}(x^2 + 2x) = 2x + 2.$$

For $x > 0$:

$$h'(x) = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}.$$

Answer:

$$\boxed{h'(x) = \begin{cases} 2x + 2 & \text{if } x < 0 \\ \frac{1}{2\sqrt{x}} & \text{if } x > 0 \end{cases}}$$

(c) Determine if $h'(0)$ exists and find its value.
Solution: Check the left-hand derivative ($x \to 0^-$):

$$\lim_{x\to 0^-} \frac{h(x) - h(0)}{x - 0} = \lim_{x\to 0^-} \frac{x^2 + 2x}{x} = \lim_{x\to 0^-} (x + 2) = 2.$$

Check the right-hand derivative ($x \to 0^+$):

$$\lim_{x\to 0^+} \frac{h(x) - h(0)}{x - 0} = \lim_{x\to 0^+} \frac{\sqrt{x}}{x} = \lim_{x\to 0^+} \frac{1}{\sqrt{x}} = +\infty.$$

Since the left-hand and right-hand derivatives are not equal, $h'(0)$ **does not exist**. **Answer**: $\boxed{h'(0) \text{ does not exist}}$.

Final Notes

The student demonstrated an excellent understanding of the material, providing clear and correct solutions to all problems. Their work is thorough, well-organized, and free of errors.

Score: 20/20