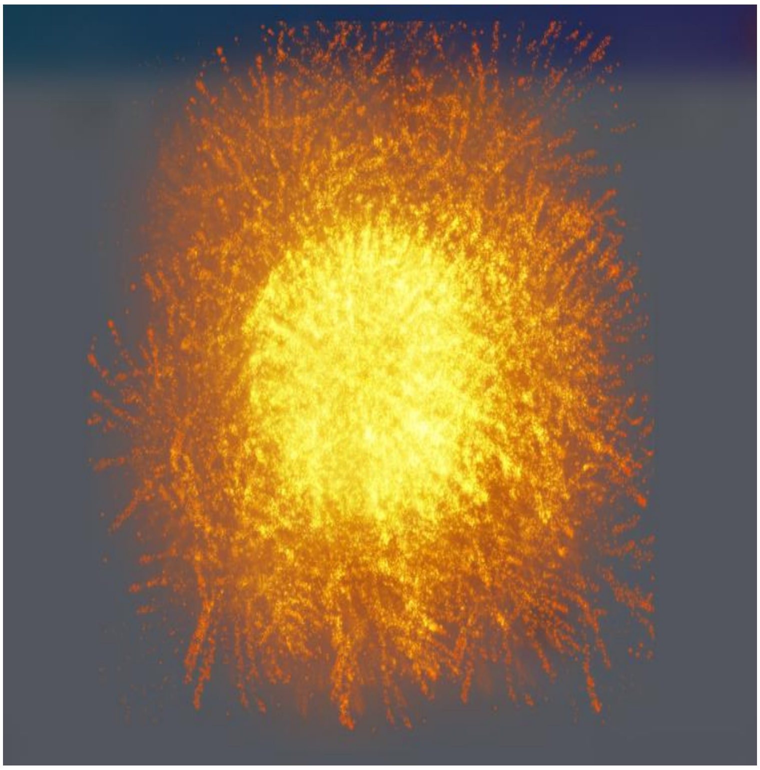
People's Democratic Republic of Algeria
Larbi Ben M'hidi University -Oum El Bouaghi-
Faculty of Sciences and Applied Sciences
Department of Mechanical Engineering

Course handout

Heat Transfer 1 — Y. Harnane

Heat Transfer 1

2nd Year Mechanical Engineering



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Heat transfer is the science that seeks to predict the energy transfer that may take place between material bodies as a result of a temperature difference. Thermodynamics teaches that this energy transfer is defined as heat. The science of heat transfer seeks not merely to explain how heat energy may be transferred, but also to predict the rate at which the exchange will take place under certain specified conditions. The fact that a heat-transfer rate is the desired objective of an analysis points out the difference between heat transfer and thermodynamics.

Thermodynamics deals with systems in equilibrium; it may be used to predict the amount of energy required to change a system from one equilibrium state to another; it may not be used to predict how fast a change will take place since the system is not in equilibrium during the process. Heat transfer supplements the first and second principles of thermodynamics by providing additional experimental rules that may be used to establish energy-transfer rates. As in the science of thermodynamics, the experimental rules used as a basis of the subject of heat transfer are rather simple and easily expanded to encompass a variety of practical situations.

As an example of the different kinds of problems that are treated by thermodynamics and heat transfer, consider the cooling of a hot steel bar that is placed in a pail of water. Thermodynamics may be used to predict the final equilibrium temperature of the steel bar–water combination. Thermodynamics will not tell us how long it takes to reach this equilibrium condition or what the temperature of the bar will be after a certain length of time before the equilibrium condition is attained. Heat transfer may be used to predict the temperature of both the bar and the water as a function of time.

Introduction to heat transfer and its position relative to thermodynamics

1.1. Introduction

It is unnecessary to emphasize the importance of studying heat transfer, as it, arising from temperature differences, plays a fundamental role in numerous natural physical processes or those related to human activity. Thus, most mechanisms involved in the creation, conversion, and dissipation of energy proceed from or depend on these exchanges.

These exchanges or transfers stem from a very simple reality: when two media at different temperatures, meaning with different energy contents, are in contact in one way or another, the one at the higher temperature yields energy to the other in the form of heat.

Thermodynamics teaches us that energy can be transferred through interactions between a system and its surroundings, in the form of heat and work. However, thermodynamics is only concerned with the initial and final equilibrium states of the system and provides no information about the nature of the interactions involved or the temporal evolution of the system between these two equilibrium states.

The goal of heat transfer analysis is to identify which transfer modes are involved during the transformation and to quantitatively determine how the temperature varies at each point of the system over time.

There are three main heat transfer mechanisms which may or may not coexist: conduction, convection, and radiation.

1.2.The Three Modes of Heat Transfer

- **Heat Transfer by Conduction** (in solids or stationary fluids).

Conduction is a heat transfer that requires a material medium—solid, liquid, or gas—but occurs without mass transfer. It originates in the agitation of the molecules or atoms constituting the medium, because during collisions between elementary particles resulting from this agitation, the more energetic ones transfer energy to the others. At the macroscopic level, this constitutes an energy diffusion process or conduction.

- **Heat Transfer by Convection**

Convection is related to a mechanism of mass transfer (liquid or gaseous flow) arising from pressure or density gradients existing within the medium, which can thereby cause energy transfer. It is therefore more of a macroscopic transport process, but one that coexists, or can coexist, with the other heat transfer modes, conduction and radiation. It is, of course, specific to fluid media.

Two types of convection are distinguished:

- **Natural Convection:** Movements are due to variations in density within a fluid subjected to a gravitational field. Density variations can be generated by temperature gradients (hot air is lighter than cold air) and/or by composition gradients.
- **Forced Convection:** The fluid motion is caused by external mechanical actions (pump, fan...).
- We speak of **Mixed Convection** when both types of convection coexist in a system.

- **Heat Transfer by Radiation**

Radiation is a mode of heat transfer resulting from the emission of electromagnetic waves by the molecules and atoms of a medium and thus corresponds to an energy loss from these particles. The process therefore does not require a material medium, but the corresponding energy absorbed by another medium can constitute a heat input for the latter.

Fundamental Laws of Heat Transfer

2-1. Definitions

- Heat transfer is determined by the evolution in space and time of the temperature, $T(x, y, z,t)$.

- The variation over time at a point $M(x, y, z)$ of the system is given by the partial derivative of $T(x, y, z,t)$ with respect to time :$\frac{\partial T}{\partial t}$

Over a time interval dt, the temperature variation at a point M will be $dT = \frac{\partial T}{\partial t} dt$

- The variation in space at a given time t is given by **the temperature gradient**:

$$\vec{\nabla} T = \overrightarrow{grad} T = \left| \begin{matrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{matrix} \right.$$

- **Heat flux**

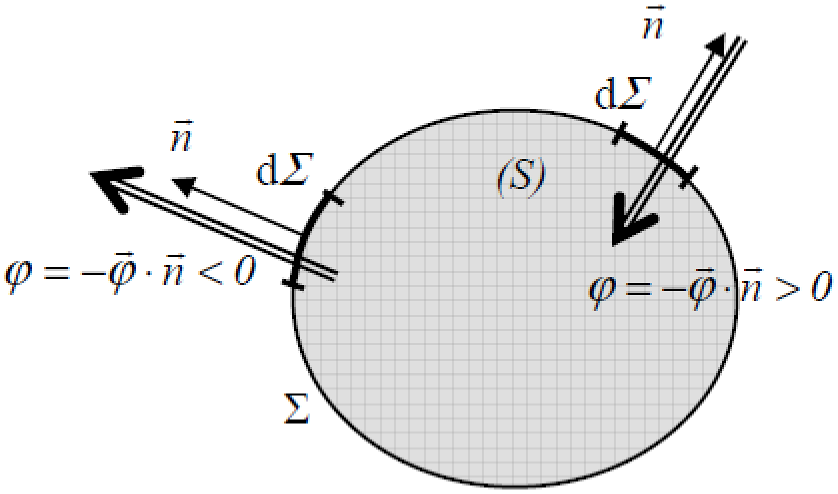
A heat flux is a quantity of energy transferred as heat per unit of time. It is therefore a power, expressed in watts (J/s).

$$\phi = \frac{Q}{t} = \dot{Q} \quad (W)$$

- **Heat flux density**

Generally, the heat flux exchanged across a surface is not uniform over the entire surface. Therefore, we define a heat flux density, φ, which corresponds to the heat flux per unit area (in W/m^2).

Example: heat flux exchanged by a system with its surroundings through a surface S:



$$\phi = \iint_{\Sigma} \varphi dS = \iint_{\Sigma} -\vec{\varphi}.\vec{n}dS$$

The vector $\vec{n}$ is the outward normal to the surface element dS.

The '-' sign is introduced to respect the following convention: flux entering the system is counted positively.

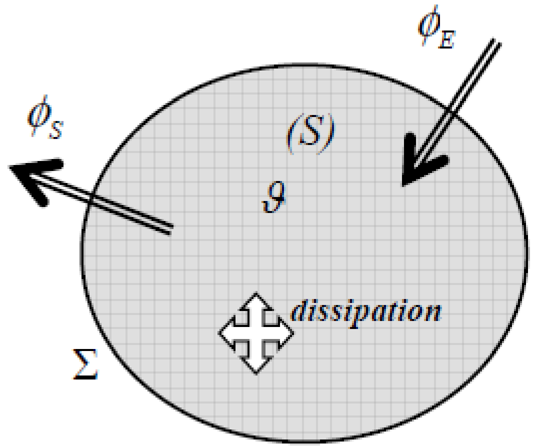
2-2. Formulation of a Heat Transfer Problem

Goal: To determine quantitatively how the temperature inside the system evolves in space and time. The equation that provides this information is called the energy equation or heat equation.

How is this equation derived?

We perform an energy balance on the system, i.e., we apply the first law of thermodynamics.

- **Step 1:** Define a control volume (v) bounded by a control surface S through which energy and matter can flow.
- **Step 2:** Identify the different energy fluxes involved that affect the state of the system. For instance, if we are interested in heat fluxes:



- Incoming heat flux : $\boldsymbol{\phi_E}$
- Outgoing heat flux : $\boldsymbol{\phi_S}$
- Heat flux dissipated (produced) in the volume : $\boldsymbol{\phi_{PR}}$

ϕ_{PR} originates from another form of energy (chemical, electrical (Joule effect), nuclear) that is converted into thermal energy inside the volume

After the exchange, the heat flux accumulated (stored) in the volume contributes to the change in internal energy, which manifests as a change in the temperature of the volume.

- **Step 3 :** Perform an energy balance by applying the first law of thermodynamics: Accumulation = Input – Output + Production

Assuming the volume is incompressible (work from pressure forces is zero) and at rest, the first law for an evolution between time *t* and *t*+*dt* is written as:

$$dU = \delta Q = \delta Q_{ext} + \delta Q_{int}$$

The term δQ accounts for both heat exchange with the surroundings δQ_{ext} (related to $\phi_E - \phi_S$), and internal heat generation, δQ_{int}, resulting from the conversion of another form of energy into heat (related to ϕ_{PR}).

$$\frac{dU}{dt} = \frac{\delta Q_{ext}}{dt} + \frac{\delta Q_{int}}{dt} = \phi_E - \phi_S + \phi_{PR}$$

- **Step 4 :** the expressions of the different fluxes are established.on établit les expressions des différents flux.

Exercises (Chap. 1 & Chap. 2)

Ex.1 :

1. How does heat transfer occur?

Some solar showers consist of a black plastic bag filled with water and placed in the sun. Identify the mode of heat transfer:

a. from the sun to the plastic bag;

b. from the plastic bag to the water it contains;

c. within the water inside the plastic bag

2. Illustrating Modes of Heat Transfer

In summer and in good weather, the water in a swimming pool is at a temperature of 25°C. The air temperature is 30°C and the temperature of the ground surrounding the pool is 17°C. In this situation, give an example where heat transfer occurs:

a. by conduction;

b. by convection;

c. by radiation

Ex.2:

A cylindrical electrical resistor (D=0.4cm, L=1.5cm) on a printed circuit board dissipates 0.6 W of power. Assuming that the heat is transferred uniformly across all surfaces, determine:

a. the amount of heat dissipated by this resistor over a 24-hour period,

b. the heat flux across the surface of the resistor,

c. the fraction of heat dissipated by the top and bottom surfaces.

Ex.3:

A bedroom window is single-glazed. The room temperature is Ti = 19 °C and the outside temperature is Te = -1 °C. These temperatures are considered constant.

1. Draw a diagram of the situation, specifying the direction of heat transfer through the glass.

2. Calculate the heat flux through the glass.

3. What is the thermal energy transferred in 1.25 h?

Data: The heat flux is given by φ = Q/Δt = |T1 – T2|/Rth. The thermal resistance of this glass is: Rth = 5.0×10^{-3} $K \cdot W^{-1}$

Ex 4 :

An electric current of intensity I is passed through a conductor of electrical resistance R and heat capacity C. The thermal power lost by the conductor is proportional to the temperature difference $\theta = T - T_0$ between the conductor and the surrounding environment. Initially, the conductor was at temperature T_0.

1. Apply the first law of thermodynamics to the system over the elementary time interval dt.
2. Deduce the temperature change over time.

Ex5 :

The heat flux density exiting a homogeneous sphere of radius R = 10 cm is uniform over its entire surface. Calculate its value if the sphere is the site of uniform heat production with the following values:

1) $\dot{q}_v = 0\ W/m^3$ 2) $\dot{q}_v = 15\ kW/m^3$

Show that isothermal surfaces are spheres. Deduce the law governing the variation of temperature inside the sphere. Calculate the temperature at its center if the surface temperature T_s is 20°C and the thermal conductivity λ is equal to 0,017*kW/m.K*.

Heat conduction

3-1. Fourier's Law

The simplest way to relate the heat flux density φ to temperature variations is to state that it is proportional to the local temperature gradient ∂T/∂r via a conductivity coefficient λ, which also depends on the temperature and the nature of the medium, i.e.,

$$\vec{\varphi} = -\lambda\vec{\nabla}T$$

This transfer mechanism is governed by a phenomenological law established by Joseph Fourier in 1822, stipulating that the heat flux density exchanged by conduction is proportional to the temperature gradient (proportionality between the cause (the gradient) and the effect (the flux)). This law is called **Fourier's Law**.

The minus sign "-" in this law indicates that the heat flux flows from hot areas to cold areas (in the opposite direction to the temperature gradient).

Notes

- Conduction results from a non-uniformity of temperature, leading to energy transfer from one point to another without macroscopic movement or transport of matter. It is a diffusion phenomenon.
- Based on the preceding definitions, the heat flux density is expressed in W/m² and the proportionality coefficient, λ, is the thermal conductivity, in W/(mK).
- Fourier's law is the analog for heat transfer of linear laws used for other types of transfer: Ohm's law for the transfer of electric charge, Fick's law for mass transfer, etc.
- Equation (3.1) is valid for an isotropic medium. In any heterogeneous (anisotropic) medium, Fourier's law is expressed using a thermal conductivity tensor. The values of the components in each principal direction of the associated diagonal matrix must then be taken into account.
- In thermodynamically balanced environments, it is generally possible to assume that internal energy depends only on temperature T. We then write that the elementary internal energy per unit mass of the medium is proportional to dT, i.e.:

$$de = CdT$$

Where C is the local specific heat capacity, which depends on the temperature and the nature of the medium.

3-2. Thermal Conductivity

Thermal conductivity depends on the nature of the medium and its physical state (primarily temperature). Qualitatively, from the most insulating media (gases) to the most conductive media (metals); the scale of λ values ranges from 10^{-1} to 10^{2} $W.m^{-1}.K^{-1}$ at typical temperatures (Table 1).

Nature of the medium	**Thermal conductivity at** 2°°C $(W.m^{-1}.K^{-1})$
Gas at atmospheric pressure	0,006-0,18
Insulating materials	0,025-0,25
Non-metallic liquids	0,1-1,0
Non-metallic solids	0,025-3
Metallic liquids	10-150
Metallic alloys	10-150
Pure metals	20-400

Table 1. Thermal Conductivity of Common Substances

The variation of λ with temperature differs greatly depending on the medium considered: gas, liquid, or solid, and can also vary significantly within each phase.

It can generally be considered that thermal conductivity varies with temperature according to the empirical correlation $\lambda = \lambda_0(1 + aT)$, where λ_0 is the thermal conductivity at 0°C and has a temperature coefficient.

Notes

For gases: (except saturated vapors) λ increases with temperature $\lambda \sim f(T^{n})$ and depends very little on pressure, with $0{,}5 < n < 1$.

For liquids: (except liquid metals) λ decreases with temperature except for water, which has a maximum conductivity around 150°C, and for liquid metals, whose conductivity is close to that of solid metals.

For homogeneous solids:

In the case of a single crystal: conductivity generally depends on the direction (anisotropic medium). In a given direction, λ increases with temperature. Maximum conductivity is reached at very low temperatures (10 to 20 K for dielectrics, a few Kelvin for metals). In the case of a polycrystal, conductivity is independent of direction. It is lower than that of a single crystal.

For homogeneous solids, such as construction materials, thermal insulators, and refractories, an apparent thermal conductivity coefficient is defined. This coefficient depends on the structure, composition, density, porosity, and moisture content.

In the case of thermal insulators: consisting of a poorly conductive gas-solid mixture, the lower the porosity and density, the lower the λ. We distinguish:

- Fibrous insulators (plant fibers, asbestos, glass wool).
- Granular insulation (granular cork, pumice stone, diatomaceous earth).
- Cellular insulation (foam).

For all three types of materials, λ increases with temperature (analogous to gases).

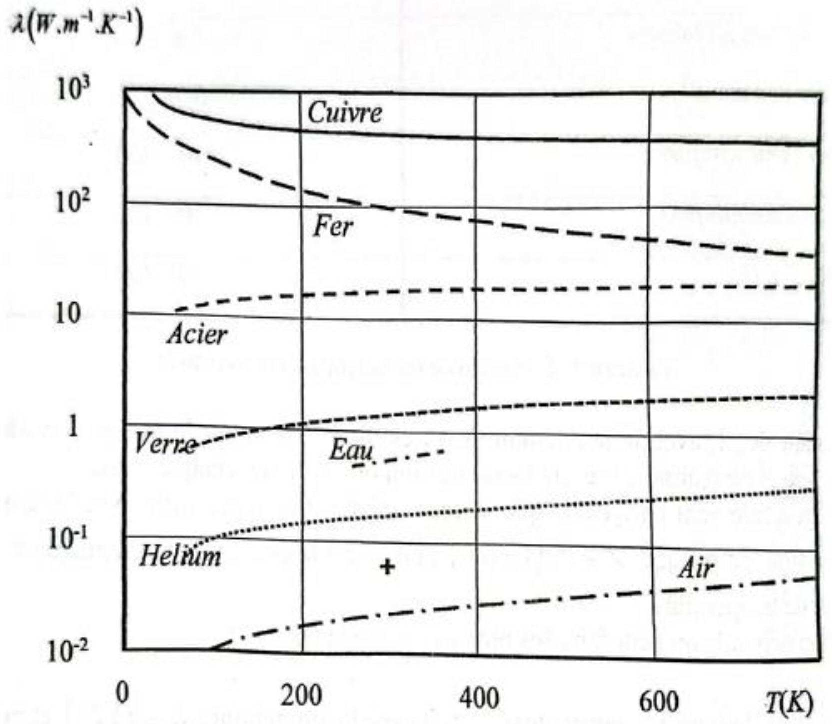


Figure 3.1: Thermal conductivity as a function of temperature (+: Glass wool)

3-3. Energy Equation

Consider a closed, homogeneous, and rigid solid (or fluid at rest) system occupying a volume (ϑ) bounded by a surface S. This system evolves over time due to energy exchanges in the form of heat with its surroundings and/or internal production of heat energy.

The temperature distribution within the volume is not uniform and changes over time. Therefore, the system is not in thermodynamic equilibrium and is subject to heat flow. To establish the equation governing the temperature evolution at each point within the volume (ϑ), we will perform an energy balance on the system. We assume that the system is at rest and that no mechanical work is involved because the system is rigid (no change in volume).

The change in internal energy of the system between times ***t*** and ***t+dt*** is then:

$$dU = \delta Q_{ext} + \delta Q_{int}$$

Where :

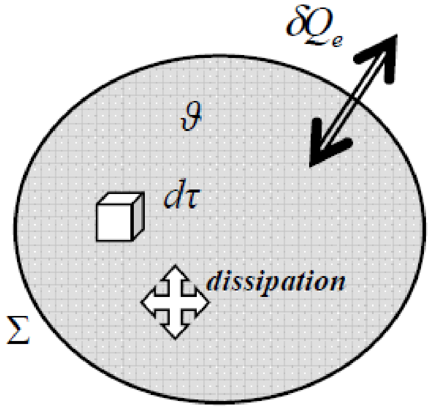
dU is the change in internal energy of the system during a time interval ***dt.***

δQ_{ext} is the amount of heat exchanged by the system with its surroundings through S during the time interval ***dt***.

δQ_{int} is the amount of heat produced by dissipation in the total volume ϑ during the time interval ***dt.***

$$\frac{dU}{dt} = \frac{\delta Q_{ext}}{dt} + \frac{\delta Q_{int}}{dt} = \phi_E - \phi_S + \phi_{PR}$$

We will therefore consider an elementary volume element, dτ, small enough that the temperature inside can be considered uniform (but large enough to contain a large number of particles). The elementary volume can then be considered in equilibrium: this is called local thermodynamic equilibrium.



System volume : $v = \iiint_{\vartheta} d\tau$

Mass contained in $d\tau$: $dm = \rho d\tau$

ρ density of the body

Mass of the system : $m = \iiint_{\vartheta} \rho d\tau$

- **Variation of internal energy of the mass m contained in (ϑ) between times *t* and t *t+dt***

The variation of internal energy per unit mass of the system is :

$$du = cdT$$

Where : u is the specific internal energy

c is the specific heat (in $J.K^{-1}.kg^{-1}$) of the material

- The variation of internal energy for the mass ***dm*** contained in the elementary volume $\boldsymbol{d\tau}$ (considered to be in thermodynamic equilibrium, therefore at uniform temperature) is :

$$dm\, du = \rho d\tau du = \rho\, d\tau\, c\, dT = \rho\, d\tau\, c\, \frac{\partial T}{\partial t} dt$$

- By integrating over the entire volume, we obtain the variation of internal energy for the mass ***m*** contained in **(ϑ)** during the time interval ***dt*** :

$$dU = dt \iiint_{\vartheta} \rho\, c\, \frac{\partial T}{\partial t} d\tau$$

Let, per unit time:

$$\frac{dU}{dt} = \iiint_{\vartheta} \rho\, c\, \frac{\partial T}{\partial t} d\tau$$

a. **Heat flux (or thermal power) dissipated inside the volume (ϑ):**

Let P be the internal volumetric thermal power generation (en W/m^3).

$$\phi_{PR} = \iiint_{\vartheta} P d\tau$$

b. **Heat flux exchanged by the system with the exterior through the surface S:**

$$\phi_E - \phi_S = \phi = \iint_{\Sigma} -\vec{\varphi}.\vec{n}\, dS$$

With $\vec{\varphi} = -\lambda \vec{\nabla} T$ (heat transfer by conduction – Fourier’s law)

$$\Rightarrow \phi_E - \phi_S = \iint_{\Sigma} \lambda \vec{\nabla} T.\vec{n} dS$$

The first principle $\frac{dU}{dt} = (\phi_E - \phi_S) + \phi_{PR}$ is written as :

$$\iiint_{\vartheta} \rho\, c\, \frac{\partial T}{\partial t} d\tau = \iint_{\Sigma} \lambda \vec{\nabla} T.\vec{n} dS + \iiint_{\vartheta} P d\tau$$

By applying the divergence theorem (Ostrogradsky's theorem or the divergence theorem :: $\forall\, \vec{V} \iint_S \vec{V}.\vec{n} dS = \iiint_{\vartheta} div\vec{V} d\tau$) to the surface integral, we obtain :

$$\underbrace{\iiint_{\vartheta} \rho c \frac{\partial T}{\partial t} d\tau =}_{accumulation} \underbrace{\iiint_{\vartheta} div(\lambda \vec{\nabla} T) d\tau}_{échangeavecl'environnement} + \underbrace{\iiint_{\vartheta} P d\tau}_{productioninterne}$$

This balance constitutes **the heat equation in global form** (integrated over the entire volume). It is valid regardless of the volume element $d\tau$. One can then write **a local heat equation**, which allows, after solving, to determine the temperature at any point in the system at each instant.

<u>Local heat equation :</u>

$$\rho\, c\, \frac{\partial T}{\partial t} = div\left(\lambda \vec{\nabla} T\right) + P \qquad \text{within } (\vartheta)$$

In the case where λ can be considered constant (homogeneous medium and λ independent of T):

$$\rho\ c\ \frac{\partial T}{\partial t} = \lambda div\left(\vec{\nabla} T\right) + P$$

$$\rho\ c\ \frac{\partial T}{\partial t} = \lambda \nabla^2 T + P \qquad \text{dans } (\vartheta)$$

$$\frac{\partial T}{\partial t} = \frac{\lambda}{\rho\ c} \nabla^2 T + \frac{P}{\rho\ c} \qquad \text{dans } (\vartheta)$$

Where

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \qquad \text{(Laplacien)}$$

$\boldsymbol{\alpha = \frac{\lambda}{\rho c}}$*(m²/s)* **the thermal diffusivity** of the medium, which quantifies the speed at which heat diffuses within the medium.

Integrating the heat equation allows obtaining ***T(x,y,z,t)*** One must specify:

- An initial condition $T(x, y, z, t = 0)$ which defines the initial thermal state of the system.
- Two boundary conditions imposed at the boundaries. These conditions can be of two types :

3-4. Boundary Conditions

The heat propagation equation is a second-order partial differential equation, requiring knowledge of spatial and temporal boundary conditions that determine the evolution of the system.

Thus, it is necessary to know the initial temperature distribution (in an unsteady problem) as well as the temperature (or its normal derivative) at the boundaries of the considered domain because transfers with the external environment occur at the surfaces delimiting this medium. They generally take place by conduction, convection, or radiation.

3-4-1. External Boundary Conditions

Three types are generally distinguished:

a. **Dirichlet conditions**: this is a condition of imposed temperature field at the separation surface. At each point $\boldsymbol{M}$ of the external boundary, the temperature $T_w(r,t)$ takes a known value $\theta(r,t)$.

This condition is encountered, for example, in the case of an insulating medium in contact with an external body of large heat capacity and high conductivity, or in the case of a phase change.

b. **Neumann conditions**: this is a condition of imposed flux at the surface. At each point M of the surface, the flux density crossing the system has an imposed value, $\varphi(r,t)$.

In practice, this condition occurs when a heat source acts directly on the surface (Joule effect, absorption of solar radiation, etc.). The particular case $\varphi(r,t) = 0$ corresponds to a perfectly insulated or adiabatic boundary.

c. Mixed or Fourier condition: this condition is based on the hypothesis of proportionality between the flux leaving or entering the solid and the difference between the surface temperature $T_w(r,t)$ and a temperature T_{eq} characteristic of the ambient medium. The proportionality coefficient is the overall surface-to-ambient heat transfer coefficient

$$-\lambda\left[\frac{\partial T}{\partial n}\right]_{(r,t)} = h\left[T_w(r,t) - T_{eq}\right]$$

This is the most commonly encountered condition in practice; it accounts for transfers by convection, radiation, or a combination of these transfers.

3-4-2. Internal Boundary Conditions

These are conditions between two media of different conductivity but where only the phenomenon of conduction is involved. In fact, the thermal properties of two continuous media do not vary discontinuously when crossing their interface.

There is always a transition layer of finite thickness where these properties evolve rapidly but continuously. The thickness of this layer can be extremely small, on the order of a few interatomic distances if these media adhere perfectly to each other.

It can be much more significant in the case of contact between two solid media, considering their surface irregularities, and their possible binder (bonding, non-autogenous welding), or simply their contact within a fluid (air, for example).

Schematic of internal boundary conditions

They allow relating temperatures and flux densities at the interface of two conductive media. In the most frequent case, that is, contact without interfacial heat generation, two conditions are considered:

a- **Perfect contact condition:** it assumes continuity of temperatures and flux densities at the interface (Fig. 3.2)

$$T_1 = T_2$$

$$\lambda_1 \left[\frac{\partial T_1}{\partial x}\right] = \lambda_2 \left[\frac{\partial T_2}{\partial x}\right]$$

b. **Imperfect contact condition:** it assumes discontinuity of temperatures but continuity of flux densities (Fig. 3.3). The difference between the extrapolated temperatures in media 1 and 2 is assumed proportional to the flux density. The proportionality coefficient R is called the thermal contact resistance (Fig. 3.3).

$$T_1 - T_2 = \Delta T = R\phi$$

$$\phi = -\lambda_1 \left[\frac{\partial T_1}{\partial x}\right] = -\lambda_2 \left[\frac{\partial T_2}{\partial x}\right]$$

3-4-3. Initial Conditions

An unsteady conduction problem is only well-posed if the thermal state of the system at the beginning of its evolution is described.

The initial condition is expressed by the knowledge of the temperature field $T(r)$ à $t = 0$ either

$$\lim_{t\to 0} T(r,t) = T(r,0) = T_0(r)$$

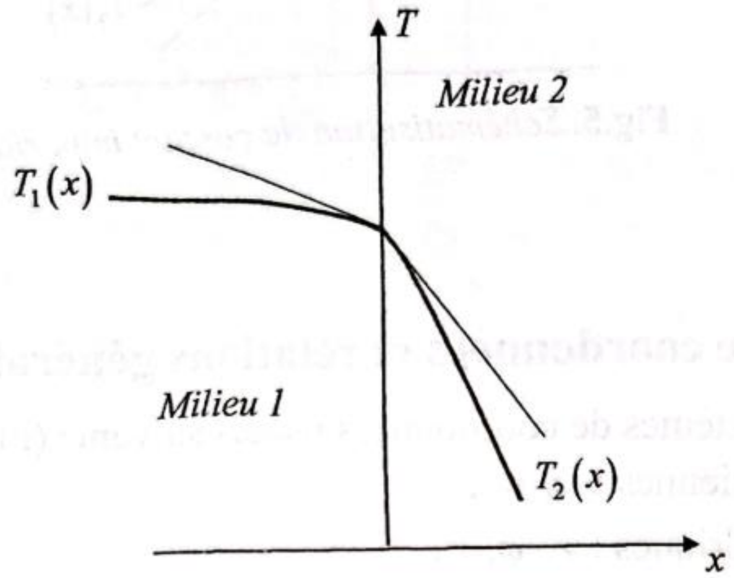


Figure 3.2. Perfect contact

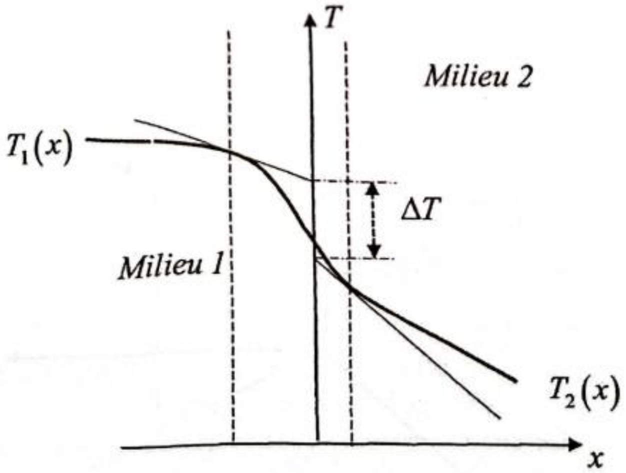


Figure 3.3. Imperfect contact

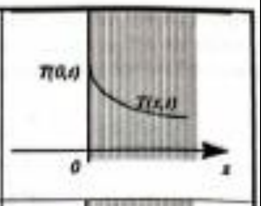
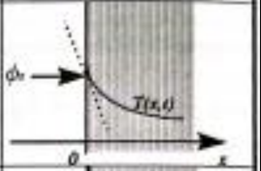
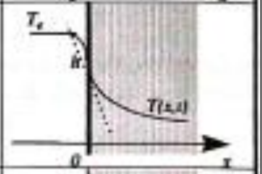
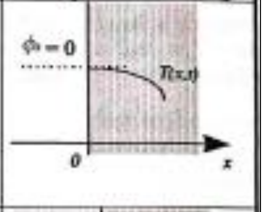
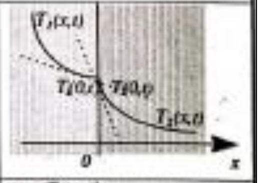
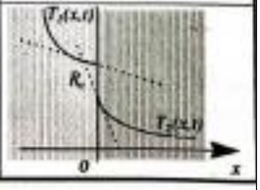
Temperature condition	$T(0,t)=T_0$	
Flow condition	$\left(-\lambda\frac{\partial T}{\partial x}\right)_{x=0}=\phi_0$	
Newton Condition (Fluid mixing temperature T_e	$\left(-\lambda\frac{\partial T}{\partial x}\right)_{x=0}=h\left[T_e-T(0,t)\right]$	
Insulated wall or symmetries imposing zero flow	$\left(-\lambda\frac{\partial T}{\partial x}\right)_{x=0}=0$	
Perfect contact	$\left(-\lambda_1\frac{\partial T_1}{\partial x}\right)_{x=0}=\left(-\lambda_2\frac{\partial T_2}{\partial x}\right)_{x=0}$ $T_1=T_2$	
Imperfect contact	$\left(-\lambda_1\frac{\partial T_1}{\partial x}\right)_{x=0}=\left(-\lambda_2\frac{\partial T_2}{\partial x}\right)_{x=0}$ $\left(-\lambda_1\frac{\partial T_1}{\partial x}\right)_{x=0}=\frac{\Delta T}{R_c}$	

Table 2. Boundary conditions in the one-dimensional case

4. Steady-state conduction without internal heat dissipation

4.1. Heat Equation

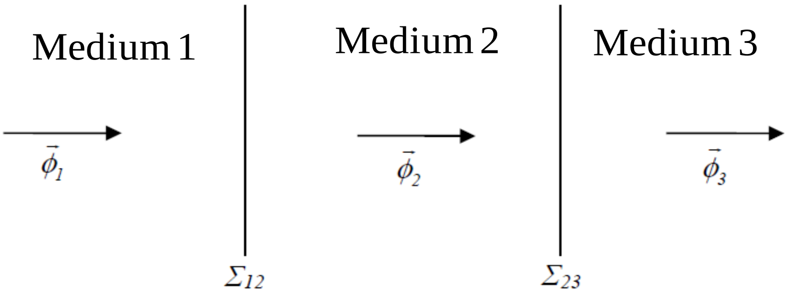
Consider a homogeneous and undeformable solid (or fluid at rest) and assume the thermal conductivity of the material is constant. The heat equation established previously:

$$\rho\, c\, \frac{\partial T}{\partial t} = \lambda \nabla^2 T + P$$

- In steady state (stationary) : $\frac{\partial T}{\partial t} = 0 \;\Rightarrow T(x, y, z)$
- Without internal heat dissipation: $P = 0 \Rightarrow \nabla^2 T = 0$ dans (ϑ)

In steady state without dissipation, the balance of flux entering and leaving the domain is zero. Therefore, there is **conservation of heat flux**:

$$\phi = \phi_E - \phi_S = \iint_S -\vec{\varphi}.\vec{n}\, dS = 0$$



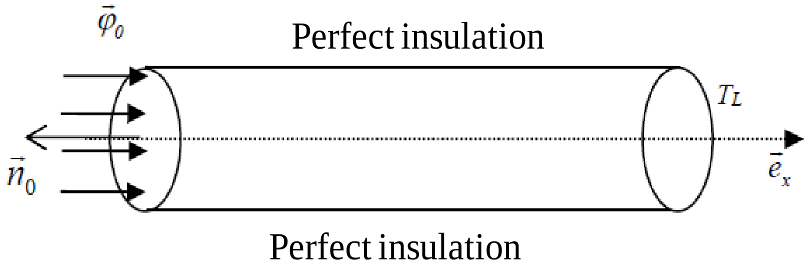
Flux conservation : $\overrightarrow{\varphi_1} = \overrightarrow{\varphi_2} = \overrightarrow{\varphi_3} = \cdots$

4-1. Applications :

a- Conduction in a bar

Consider a cylindrical bar of length ***L*** and cross-section ***S***, made of a homogeneous material with thermal conductivity λ assumed constant. This bar is heated at one end by Joule effect and cooled at the other end to a given temperature (for example, by circulating a cooling liquid). Assume steady state is reached.

Assume the bar is perfectly insulated on its lateral surface (therefore no heat exchange with the exterior through this surface). We will thus be able to assume that the heat flux propagates only in the axial direction $\overrightarrow{e_x}$ (one-directional flux).



The temperature inside the bar then depends only on one spatial variable x: $T = T(x)$

The heat equation is written:

$$\nabla^2 T = \frac{d^2T}{dx^2} = 0$$

After integration:

$$\frac{dT}{dx} = Cst = A$$

$$\Leftrightarrow \qquad T(x) = Ax + B$$

The temperature distribution inside the bar is therefore linear.

Determining the 2 constants A and B requires knowledge of 2 boundary conditions.

- Determination of A: at $x = 0$, a heat flux $\phi_0 (T(x = 0) = T_0$ unknown) is imposed

$\phi_0 = \iint_S -\overrightarrow{\varphi_0}.\vec{n}dS = \iint_S \overrightarrow{\varphi_0}.\overrightarrow{e_x}\,dS$

with

$$\overrightarrow{\varphi_0} = -\lambda \vec{\nabla} T|_{x=0} = -\lambda \left.\frac{dT}{dx}\right|_{x=0} \overrightarrow{e_x}$$

and

$$\left.\frac{dT}{dx}\right|_{x=0} = A \quad \Rightarrow \phi_0 = \iint_S -\lambda \left.\frac{dT}{dx}\right|_{x=0} dS = \iint_S -\lambda\, A\, dS = -\lambda\, S\, A \quad \Rightarrow$$

$$A = \frac{-\phi_0}{\lambda S}$$

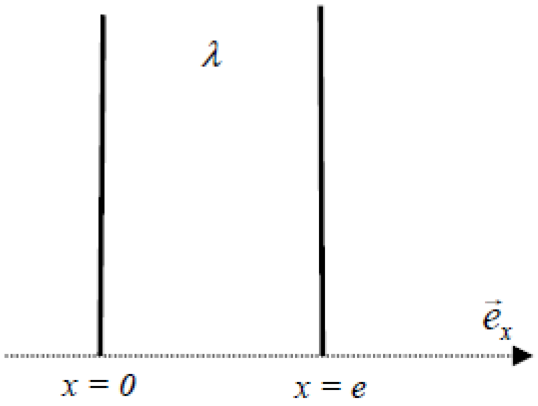
- Determination of B : at x=L, a temperature $T(x = L) = T_L$ is imposed:

$$\Rightarrow \quad T(x = L) = \frac{-\phi_0}{\lambda S} L + B = T_L \quad \Rightarrow \quad B = \frac{\phi_0}{\lambda S} L + T_L \quad \Rightarrow \quad T(x) - T_L = \frac{\phi_0}{\lambda S}(L - x)$$

One can then determine the temperature of the bar at x = 0 : $T_0 - T_L = \frac{\phi_0}{\lambda S} L$

b. The wall problem

Consider the classic problem of the wall, i.e., a homogeneous and isotropic medium, without internal heat source, with constant conductivity λ and bounded by two parallel plane walls at constant temperatures respectively. Assume the height and depth of the wall are very large compared to its thickness so that the one-dimensional problem hypothesis can be made.

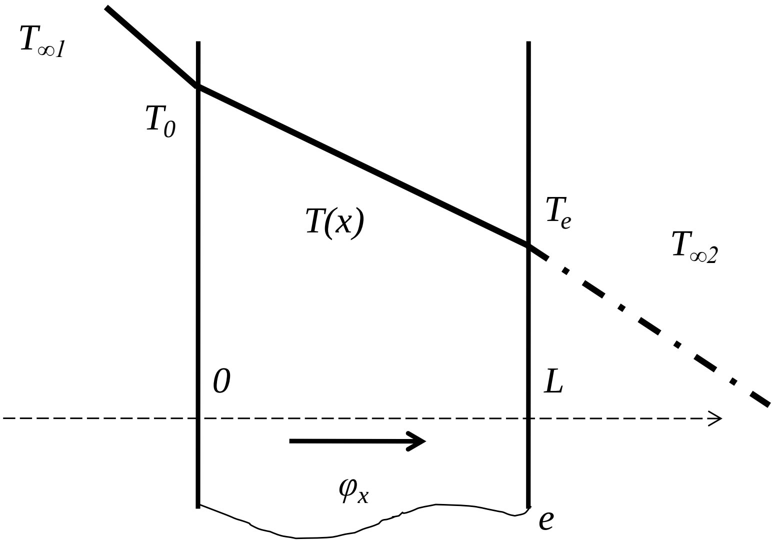


The heat flux will propagate in a single direction (along x, for example) and the temperature inside the wall will depend only on one variable:

$$T = T(x)$$

$$\Rightarrow \quad \nabla^2 T = \frac{d^2 T}{dx^2} = 0$$

$$\Leftrightarrow \quad \frac{dT}{dx} = Cst = A \quad \Leftrightarrow \quad T(x) = Ax + B$$



The temperature distribution inside the wall is linear. If we denote:

$T_0 = T(x = 0)$ et $T_e = T(x = e)$

$$\Rightarrow \quad T(x) = \frac{T_e - T_0}{e} x + T_0 \qquad (1)$$

$$\Leftrightarrow \quad \theta(X) = \frac{T(x) - T_0}{T_e - T_0} = X$$

Where $X = \frac{x}{e}$

Applying Fourier's law, we can determine the flux density crossing the wall at any x :

$$\vec{\varphi} = -\lambda \vec{\nabla} T = -\lambda \frac{dT}{dx} \vec{e}_x = \lambda \frac{T_0 - T_e}{e} \vec{e}_x = \varphi_x \vec{e}_x$$

Where we set:

$$\varphi_x = \lambda \frac{T_0 - T_e}{e} \quad \text{en} \quad (\mathrm{W/m^2})$$

$\varphi_x > 0 \ (si \ T_0 > T_e)$ or $\varphi_x < 0 \ (si \ T_0 < T_e)$

The heat flux crossing the wall surface for any x is written:

$$\phi_x = \iint_S \varphi_x dS = \lambda S \frac{T_0 - T_e}{e} \quad \text{in} \quad (\mathrm{W})$$

$$\phi_x > 0 \ ou \ < 0$$

Where

S = height × depth of the wall = surface area of the wall crossed by the heat flux

Note that the heat flux does not depend on *x*, which implies, in particular, that the flux crossing the boundary at *x=0* will be equal to the flux crossing the boundary at *x=e*, thus verifying flux conservation in the case of steady state without dissipation.

The previous relation can be written as:

$$\phi = \phi_x = \frac{T_0 - T_e}{\frac{e}{\lambda S}} \qquad (2)$$

and

$$T(x) = -\frac{\phi}{\lambda S}x + T_0 \qquad (3)$$

If both faces of the wall are at imposed temperatures

Then equation (1) completely determines the temperature distribution. The heat fluxes at the boundaries are a priori unknown, but in steady state without dissipation, the heat flux is conserved and therefore does not depend on x : $\overrightarrow{\phi_0} = \overrightarrow{\phi_e} = \vec{\phi}$, ϕ, given by equation (2).

Fluxes at the boundaries from the wall's point of view:

$\varphi = -\vec{\varphi}.\vec{n}$ where $\vec{n}$ is the outward normal to the considered boundary.

In this problem : $\vec{\varphi} = -\lambda \frac{dT}{dx}\overrightarrow{e_x} = \lambda \frac{T_0 - T_e}{e}\overrightarrow{e_x}$ independant of *x*

- at $x = 0: \vec{n} = -\overrightarrow{e_x}$ ⇒ $\varphi_0 = \overrightarrow{\varphi}|_{x=0}.\vec{e}_x = \lambda \frac{T_0 - T_e}{e}$ in (*W/m²)*

We verify that φ_0 is positive (entering the wall) when $T_0 > T_e$ (heat flows from hot to cold).

- at $x = e: \vec{n} = \overrightarrow{e_x}$ ⇒ $\varphi_e = \overrightarrow{\varphi}|_{x=e}.\vec{e}_x = -\lambda \frac{T_0 - T_e}{e} = \lambda \frac{T_e - T_0}{e}$ in (*W/m²)*

We verify that φ_e is negative (leaving the wall) when $T_0 > T_e$

If at least one of the wall faces is in contact with a flowing fluid

The wall problem with Neumann conditions amounts to imposing a flux density on one of the faces, because, in the absence of heat sources (or sinks), the flux is found on the other face. It is thus assumed that at:

- The boundary at *x = 0* is maintained at constant temperature, T_0.

- The boundary at $x = e$ is subjected to a convective flux due to the fluid flow, characterized by the convective heat transfer coefficient, h. The fluid temperature far from the wall is known, equal to T_∞

The temperature distribution in the wall will still be given by equation (1), but in this case, the temperature T_e is unknown because it results from heat exchange by conduction inside the wall and by convection with the fluid. Similarly, the heat flux in the wall is given by equation (2), but here again T_e must be known to calculate it.

<u>At the boundary *x*=*e* :</u>

- The heat flux on the wall side ($x=e^-$) is given by equation (2) (Fourier's law) :

$$\phi|_{x=e^-} = \lambda S \frac{T_0 - T_e}{e}$$

- The heat flux exchanged by convection in the fluid ($x=e^+$) is given by Newton's law :

$$\phi|_{x=e^+} = hS(T_e - T_\infty)$$

Continuity of heat flux at the solid fluid interface (at ***x*=*e***) imposes:

$$\phi|_{x=e^-} = \phi|_{x=e^+} = \phi \iff \qquad \lambda S \frac{T_0 - T_e}{e} = hS(T_e - T_\infty) = \phi$$

We can express T_e as a function of the problem data :

$$\iff \quad \left(h + \frac{\lambda}{e}\right) T_e = \frac{\lambda}{e} T_0 + hT_\infty$$

Let us determine the heat flux crossing the system:

- A first method consists of substituting the expression for T_e into the expression for ϕ
- A second method, much more widely used, allows avoiding the explicit calculation of T_e

We have seen that: $\lambda S \frac{T_0 - T_e}{e} = hS(T_e - T_\infty) = \phi$ i.e :

$$\begin{cases} T_0 - T_e = \dfrac{e}{\lambda S}\phi \\ T_e - T_\infty = \dfrac{1}{hS}\phi \end{cases}$$

By adding these two relations:

$$T_0 - T_\infty = \left(\frac{e}{\lambda} + \frac{1}{h}\right)\varphi = \left(\frac{e}{\lambda} + \frac{1}{h}\right)\frac{\phi}{S}$$

The heat flux through the wall is:

$$\phi = \frac{T_0 - T_\infty}{\left(\frac{e}{\lambda S} + \frac{1}{hS}\right)} \qquad \text{en [W]} \qquad (4)$$

Thus, we obtain an expression for the heat flux based on the problem data, without needing to calculate the temperature T_e, which is unknown a priori. The temperature distribution within the wall can then be determined using equation (3):

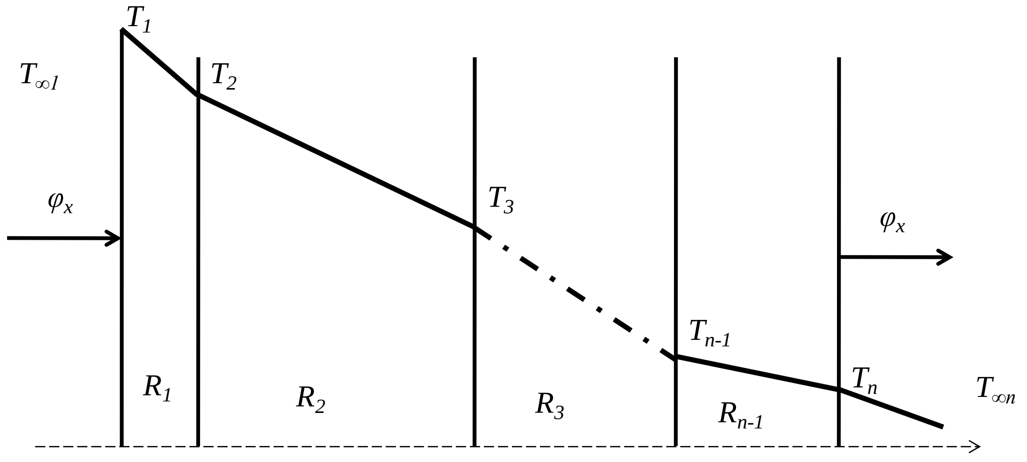
$$T(x) - T_0 = -\frac{\phi}{\lambda S}x$$

To find the temperature at the boundary $x = e$

$$T(x = e) - T_0 = -\frac{\phi}{\lambda S}e$$

- **Composite wall**

We consider (n-1) layers of materials with respective thicknesses e_1, e_2, ……, e_{n-1}, and thermal conductivities λ_1, λ_2, ……, λ_{n-1}. The temperatures on the faces are respectively, T_1, T_2, ……, T_n (Figure below) with $T_1 > T_n$.



If there is no heat loss or heat source, the same heat flux flows through all layers, and we have:

$$\phi = \frac{\lambda_1 S}{e_1}(T_1 - T_2) = \frac{\lambda_2 S}{e_2}(T_2 - T_3) = \cdots = \frac{\lambda_{n-1} S}{e_{n-1}}(T_{n-1} - T_n) \qquad (5)$$

In another form:

$$\phi = \frac{(T_1-T_2)}{R_1} = \frac{(T_2-T_3)}{R_2} = \cdots = \frac{(T_{n-1}-T_n)}{R_{n-1}} \qquad (6)$$

We can also write the global relation:

$$\phi = \frac{(T_1-T_n)}{R} \qquad (7)$$

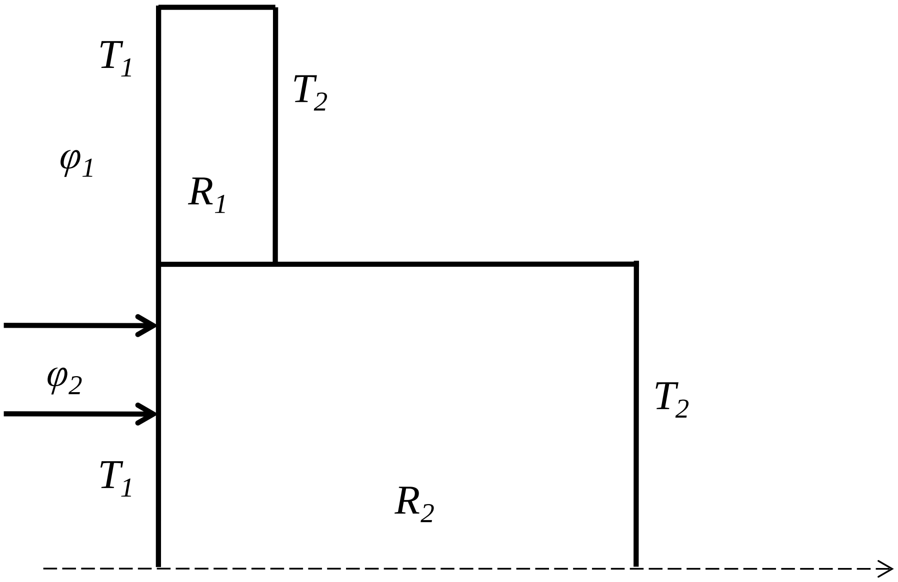
where R is the total resistance. Thus, from equations (6) and (7)

$$(T_1 - T_n) = R\phi$$

$$(T_1 - T_n) = (T_1 - T_2) + (T_2 - T_3) + \cdots \ldots .. + (T_{n-1} - T_n) = (R_1 + R_2 + \cdots . + R_{n-1})\phi$$

$$R = \sum_{i=1}^{i=n-1} R_i \qquad (8)$$

In the case of walls in parallel, for example, consisting of superimposed layers with different thicknesses and conductivities, but with the same temperature on each face.



In the case of a wall with two layers in parallel, we have:

$$\phi = \phi_1 + \phi_2 = \frac{T_1-T_2}{R_1} + \frac{T_1-T_2}{R_2} \qquad (9)$$

Hence,

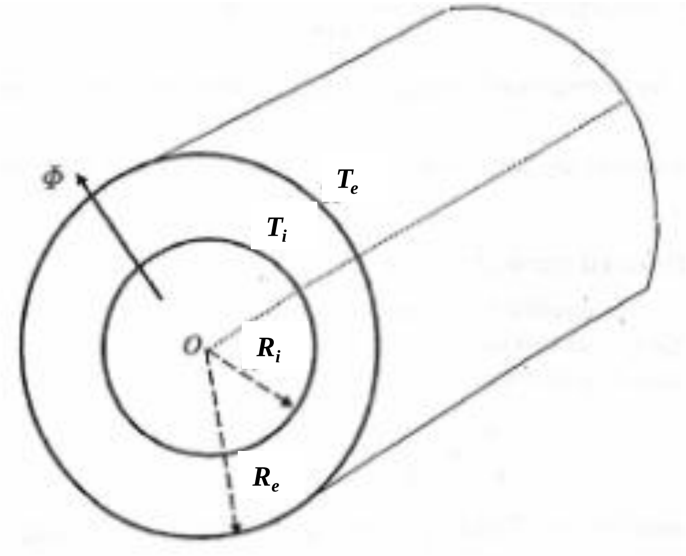
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

This can be generalized to the case of n walls in parallel, namely:

$$\frac{1}{R} = \sum_{i=1}^{i=n} \frac{1}{R_i} \qquad (10)$$

a- Conduction in cylindrical geometry

A simple and common example of cylindrical geometry is a hollow cylinder (or tubular pipe) that is sufficiently long compared to the radii R_i (inner radius) and R_e (outer radius). Thus, the parameters depend only on the radial distance r.



In the absence of a heat source and with constant thermal conductivity, the conduction equation is simply written as:

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \qquad (11)$$

With the boundary conditions:

$$\begin{cases} T = T_i & (inner\ cylinder) \\ T = T_e & (outer\ cylinder) \end{cases}$$

Equation (11) is integrated to give the following temperature profile in the annular space:

$$T = \frac{T_e - T_i}{\ln(\frac{R_e}{R_i})} lnr - \frac{T_e lnR_i - T_i lnR_e}{\ln(\frac{R_e}{R_i})} \qquad (12)$$

Note: As with the plane wall, the problem can be easily handled with other boundary conditions.

For the heat flux over a cylinder length L, we have:

$$Q = -\lambda(2\pi Lr)\frac{dT}{dr}$$

Hence,

$$\phi = \frac{2\pi L\lambda}{ln\left(\frac{R_e}{R_i}\right)}(T_i - T_e) \qquad (13)$$

The thermal resistance of the annular part is:

$$R_{th} = \frac{T_i - T_e}{\phi} = \frac{ln\left(\frac{R_e}{R_i}\right)}{2\pi L\lambda} \qquad (14)$$

b- Conduction in spherical geometry

The conduction equation in spherical coordinates (without heat source, constant thermal conductivity) for a hollow sphere with outer radius R_e and inner radius R_i is written as:

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \qquad (15)$$

With Dirichlet boundary conditions:

$$\begin{cases} T = T_i & (at\ r = R_i\ ) \\ T = T_e & (at\ r = R_e\ ) \end{cases}$$

The following temperature distribution is obtained:

$$T = T_i + \frac{T_i - T_e}{r_i^{-1} - r_e^{-1}}\frac{1}{r} - \frac{T_i - T_e}{1 - \frac{r_i}{r_e}} \qquad (16)$$

The heat flux is therefore:

$$\phi = \frac{4\pi\lambda}{r_i^{-1} - r_e^{-1}}(T_i - T_e) \qquad (17)$$

The corresponding thermal resistance is thus equal to:

$$R_{th} = \frac{r_i^{-1} - r_e^{-1}}{4\pi\lambda} \qquad (18)$$

5. Steady-state conduction with heat source

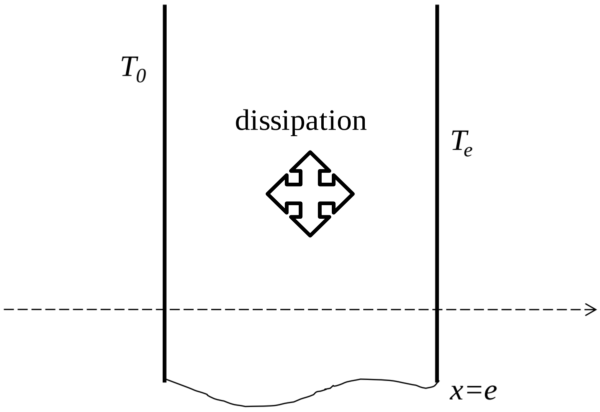
In this case, we consider conduction with an internal heat source (or sink) in simple geometries, under steady-state conditions and in a medium with constant physical properties.

a. Plane wall :

We consider the "wall" problem with a heat source assumed to be uniformly distributed.

The conduction equation is written as:

$$\lambda \frac{d^2T}{dx^2} + P = 0 \qquad (19)$$



For a wall of thickness $2e$ with faces at temperatures T_0 and T_e respectively (Figure **), integration of equation (***) gives the following profile:

$$T(x) = \frac{-P}{2\lambda}(x^2 - ex) + (T_e - T_0)\frac{x}{e} + T_0 \qquad (20)$$

The distribution is parabolic; the heat flux is not constant and varies linearly with x, namely:

$$\phi(x) = PS\left(x - \frac{e}{2}\right) + \lambda S \frac{(T_0 - T_e)}{e} \quad [\mathrm{W}] \qquad (21)$$

Where S is the wall surface area perpendicular to the heat flux.

In the case where $T_0 = T_e$

$$\phi(x) = PS\left(x - \frac{e}{2}\right)$$

At $x = 0$ we have : $\phi(x = 0) = -PS\frac{e}{2} < 0$

At $x = e$ we have : $\phi(x = e) = PS\frac{e}{2} > 0$

b. Problems with rotational symmetry

For this type of problem, one generally uses a cylindrical coordinate system. The Laplacian operator is then expressed as:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

In problems with rotational symmetry:

$$\frac{\partial}{\partial \theta} = 0$$

Furthermore, in the problems addressed in this course, it will be assumed that $\frac{\partial}{\partial z} = 0$, which amounts to reducing the problem to a one-directional (or one-dimensional) problem where the heat flux propagates only in the $.\overrightarrow{e_r}.$ direction (radial flux) and the temperature depends only on r: $T(r,\theta,z) = T(r)$.

We consider a cylinder made of a homogeneous, non-deformable material with constant thermal conductivity. It is assumed that the height of the cylinder is very large compared to its diameter, so that the one-dimensional problem hypothesis can be applied.

The heat equation in the absence of dissipation then becomes:

$$\nabla^2 T = \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \qquad \Longrightarrow \; r\frac{dT}{dr} = A \qquad \frac{dT}{dr} = \frac{A}{r} \Longrightarrow T(r) = Aln(r) + B$$

The constants A and B are to be determined, which requires two boundary conditions. **Solid cylinder of radius R:** $0 \leq r \leq R$

For $r \to 0,\ \ln(r) \to \infty \Longrightarrow T(r \to 0) \to \infty$, which is physically impossible.

$T(r) = B = T(R)$ $\qquad \forall\, r \Longrightarrow$ Isothermal cylinder

Hollow cylinder with inner radius R₁ and outer radius R₂

$T(r) = Aln(r) + B$ with $R_1 \leq r \leq R_2$

Let T_1 and T_2 be the temperatures on the inner and outer surfaces of the cylinder."

$$\begin{cases} T_1 = T(R_1) = Aln(R_1) + B \\ T_2 = T(R_2) = Aln(R_2) + B \end{cases}$$

$$\Longrightarrow\ A = \frac{T_2 - T_1}{\ln(\frac{R_2}{R_1})} \qquad \text{and} \qquad B = \frac{T_1\ln(R_2) - T_2\ln(R_1)}{\ln(\frac{R_2}{R_1})}$$

$T(r) = \frac{T_2 - T_1}{\ln(\frac{R_2}{R_1})} \ln\left(\frac{r}{R_1}\right) + T_1$ (1b)

Let us determine the flux density passing through the cylinder at any given 'r' by applying Fourier's law.

$$\vec{\varphi} = -\lambda\vec{\nabla}T = -\lambda\frac{dT}{dr}\vec{e_r} = \varphi_r\vec{e_r}$$

$\Leftrightarrow \ \varphi_r = \vec{\varphi}.\vec{e_r} = -\lambda\frac{dT}{dr} == -\lambda\frac{A}{r}$ in $[W/m^2]$

$\varphi_r > 0$ or $\varphi_r < 0$

The flux density depends on 'r', unlike in the case of the plane wall.

The heat flux passing through a height H of the cylinder at any given r is:

$\phi = \iint_S \ \varphi_r ds$ $[W]$

With $ds = rd\theta dz \Longrightarrow \ \phi = \int_0^{2\pi}\int_0^{H}\varphi_r.r.d\theta.dz = r\varphi_r 2\pi H = -2\pi H\lambda A$

$$\phi = 2\pi H\lambda.\frac{T_1 - T_2}{\ln(\frac{R_2}{R_1})}$$

We observe that the heat flux does not depend on 'r', which implies in particular that the flux crossing the boundary at r = R_1 is equal to the flux crossing the boundary at r = R_2, thus verifying flux conservation in the steady-state case with no dissipation.

The flux density depends on 'r' because the surface area crossed by the flux depends on 'r'. (To ensure conservation of the total flux, the flux density will therefore be higher at r = R_1 than at r = R_2.)

The previous relation can be written as:

$\phi = \frac{T_1 - T_2}{\ln(\frac{R_2}{R_1})\big/_{2\pi H\lambda.}}$ (2b)

and (1b) can be written in the form:

$T(r) = -\frac{\phi}{2\pi H\lambda}\ln\left(\frac{r}{R_1}\right) + T_1$ (3b)

If the inner and outer surfaces of the cylinder are subjected to imposed temperatures, then combining equations (2b) and (3b) completely determines the temperature distribution inside the wall, and the heat fluxes at the boundaries will be calculated using equation (2b).

The fluxes at the boundaries from the wall's perspective:

At $r = R_1$: $\vec{n} = -\overrightarrow{e_r} \quad \Longrightarrow \quad \vec{\varphi}_{R_1} = -\vec{\varphi}|_{r=R_1} . \vec{n} = \vec{\varphi}_{R_1} . \vec{e}_r$

With

$$\vec{\varphi}|_{r=R_1} = -\lambda \vec{\nabla} T\Big|_{r=R_1} = -\lambda \frac{dT}{dr}\Big|_{r=R_1} . \vec{e}_r$$

$$\vec{\varphi}|_{r=R_1} = -\lambda \frac{dT}{dr}\Big|_{r=R_1} = -\lambda \frac{A}{R_1}$$

We can verify that φ_{R_1} is positive (incoming) when $T_1 > T_2$ (heat propagates from hot to cold).

$A = \frac{T_2 - T_1}{\ln(\frac{R_2}{R_1})} \quad (A < 0 \Rightarrow -\lambda \frac{A}{R_1} > 0)$

At $r = R_2$: $\vec{n} = \overrightarrow{e_r} \quad \Longrightarrow \quad \vec{\varphi}_{R_2} = -\vec{\varphi}|_{r=R_2} . \vec{n} = \vec{\varphi}_{R_2} . \vec{e}_r$

With

$$\vec{\varphi}|_{r=R_2} = -\lambda \vec{\nabla} T\Big|_{r=R_2} = -\lambda \frac{dT}{dr}\Big|_{r=R_2} . \vec{e}_r$$

We can also verify that φ_{R_2} is negative (outgoing) when $T_1 > T_2$.

$$\vec{\varphi}|_{r=R_2} = -\lambda \frac{dT}{dr}\Big|_{r=R_2} = -\lambda \frac{A}{R_2}$$

Thermal resistance:

Based on the results established previously, it can be seen that the expressions for the heat fluxes that pass through a medium by conduction or are exchanged by convection can be written in the form:

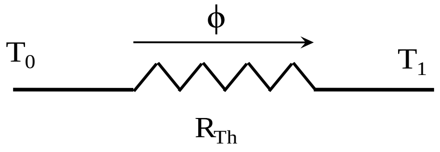
$$\phi = \frac{\Delta T}{R_{thermal}}$$

- For the plane wall: $\phi = \frac{T_0 - T_e}{\frac{e}{\lambda S}} \Rightarrow R_{th} = \frac{e}{\lambda S}$
- For the hollow cylinder: $\phi = \frac{T_1 - T_2}{\ln(\frac{R_2}{R_1}) / 2\pi H\lambda.} \Rightarrow R_{th} = \ln(\frac{R_2}{R_1}) / 2\pi H\lambda.$
- For the convective heat flux: $\phi = h.S.(T_p - T_\infty) \Rightarrow R_{th} = \frac{1}{hS}$

Thermal resistance thus represents the resistance of a medium subjected to a given temperature difference to allowing heat flux to propagate. For a given ΔT, the heat flux passing through the medium will be smaller as the resistance is larger. For example: in the case of a wall, it can be seen that the more insulating the medium (low λ), the greater the resistance, and therefore the smaller the heat flux. When dealing with a thermal insulation problem, one will therefore seek to increase the resistance of the system. When seeking to improve heat transfer (cooling of systems, heat exchangers, etc.), one will aim to reduce the resistance of the system (by increasing h, for example).

Electrical analogy:

The expression of the flux written in this way presents a certain analogy with Ohm's law in electricity: I = U/R. The heat flux plays the role of electric current, and the temperature difference that gives rise to the heat flux plays the role of the potential difference that gives rise to electric current. Thus, to represent a thermal problem, one can adopt the method of equivalent electrical diagrams of the type



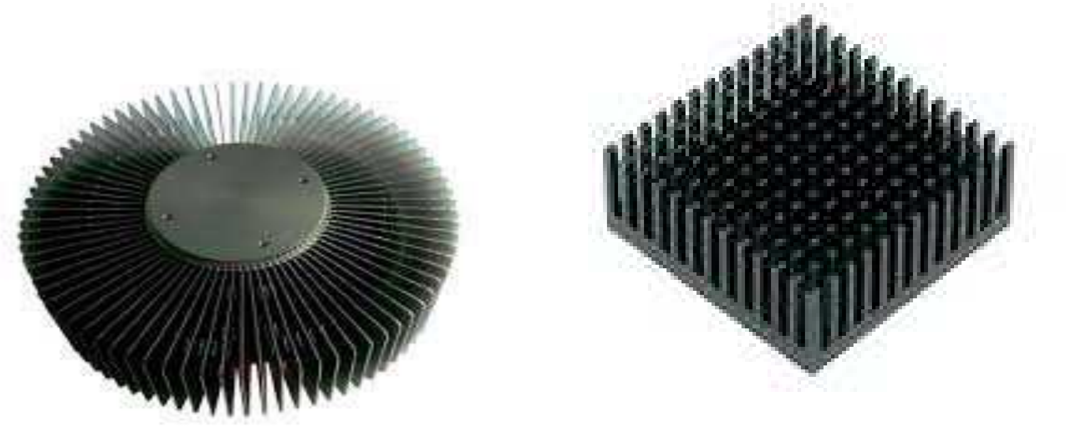
One can also apply the same combination laws as in electricity (series or parallel circuits) when dealing with a system involving multiple media and multiple types of fluxes.

6. The problem of the longitudinal rectangular fin

One of the simplest ways to increase heat transfer between a conductive medium and a fluid is to extend the exchange surface in the form of protrusions of the conductive material in a preferred direction. These are called fins and can take various shapes depending on the application (radiators, heat exchangers, etc.).

Finned heating elements

Finned tube (radiator)



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Heat sinks.

Motorcycle engine

They are generally thin enough that transverse conductance can be neglected compared to local convective heat exchange, which corresponds to situations where the ratio $Bi = \frac{hR}{\lambda}$ (the Biot number, in which R represents a characteristic transverse dimension) is small.

One can then assume that the temperature is uniform across any cross-section and that conduction is significant only in the main direction of the fin Ox, while convection occurs along its entire length.

6-1. The heat equation for a general fin

For a "thin" fin of length L and cross-sectional area *S(x)*, receiving a heat flux ϕ_0 at the base (*x=0*) , the energy balance for the elementary volume element *Sdx* is written as:

$$\frac{d}{dx}(S\phi dx) = -d\sigma h(T - T_\infty) \quad (1a)$$

Where:

$d\sigma = pdx$ Represents the elementary surface element of the fin, and $p = p(x)$ is the perimeter of the cross-section S.

Since $\phi = -\lambda \frac{dT}{dx}$ equation (1a) becomes:

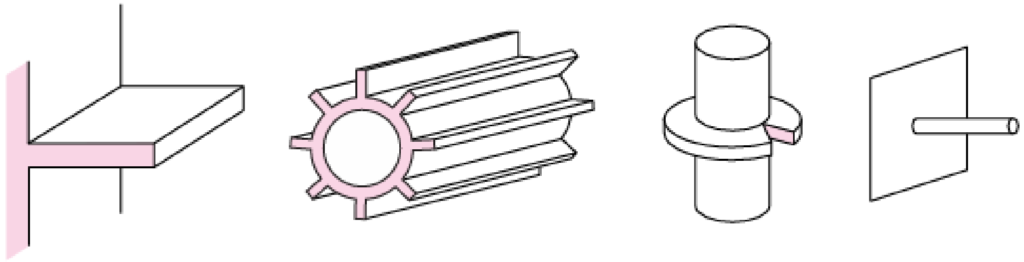
$$\frac{d^2T}{dx^2} + \frac{1}{S}\frac{dS}{dx}\frac{dT}{dx} - \frac{h}{\lambda S}\frac{d\sigma}{dx}(T - T_\infty) = 0 \quad (2a)$$

Depending on the prescribed boundary conditions, equation (1a) allows determination of the temperature profile in the fin as well as the heat flux.

6-2. Fins of uniform cross-section

6-2-1. The heat equation

The simplest fins are those with uniform rectangular or circular cross-section, Figure below:



Equation (1a) simplifies and we obtain:

$$\frac{d^2T}{dx^2} - \frac{hp}{\lambda S}(T - T_\infty) = 0 \qquad (3a)$$

Where the cross-section S and the perimeter p are constants.

Setting $\theta = T - T_\infty$ and $m^2 = \frac{hp}{\lambda S}$ equation (3a) becomes

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad (4a)$$

At the base of the fin : $x = 0$ we have $T = T_0$ hence $\theta = \theta_0 = T_0 - T_\infty$

At the tip of the fin : $x = L$ several cases are possible :

- Convective heat transfer, i.e.

$$hS(T_L - T_\infty) = -\lambda S\left(\frac{dT}{dx}\right)_L$$

$$h\theta_L = -\lambda\left(\frac{d\theta}{dx}\right)_L$$

Integration of equation (4a) then leads to the following temperature distribution:

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \qquad (5a)$$

$$\frac{d\theta}{dx} = m(C_1 e^{mx} - C_2 e^{-mx})$$

The constants C_1 and C_2 are determined from the boundary conditions at the base and at the tip of the fin.

- At the base of the fin : $x = 0$ we have $T(0) = T_0$

  hence $\theta(0) = \theta_0 = T_0 - T_\infty$

$\theta(0) = \theta_0 = C_1 + C_2$ (6a)

At the tip of the fin : $x = L$

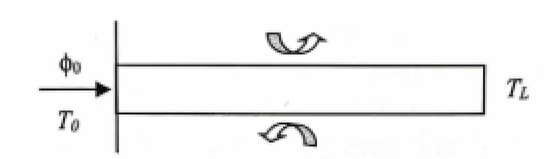
- Imposed temperature at the tip (Dirichlet type boundary condition) :

$$\theta(L) = (T(L) - T_\infty) = T_L - T_\infty = \theta_L$$

When the fin is infinitely long : $T(L) \approx T_\infty \Rightarrow \theta(L) \approx 0$

- Fin with convective heat flux at its tip (Neumann-type boundary condition) for example a thermally insulated fin (adiabatic condition).

6-2-2.Heat flux dissipated by an infinitely long fin of uniform cross-section



For a fin of "infinite" length, the temperature at the tip will be equal to the temperature of the surrounding medium, T_∞. Thus:

$\theta_L \to 0$ if $L \to \infty$

The boundary condition at $x = L$ is then written as:

$\theta(L) = \theta_L = C_1 e^{mL} + C_2 e^{-mL} \to\to_{L\to\infty} 0$ (7a)

$\Leftrightarrow C_1 \longrightarrow 0$

The boundary condition at $x = 0$ (equation (*)) then gives: $C_2 = \theta_0$

The temperature field inside the fin is given by equation (5a) with $C_1 = 0$ and $C_2 = \theta_0$

$\frac{\theta(x)}{\theta_0} = e^{-mx}$ (8a)

❖ The heat flux dissipated from the solid by the fin, ϕ_0, becomes:

$$\phi_0 = -\lambda S\left(\frac{d\theta}{dx}\right)_{x=0} = -\lambda S\theta_0\left(\frac{\frac{d\theta}{\theta_0}}{dx}\right)_{x=0} = \lambda S m\theta_0$$

Substituting m with its expression$\left(m = \sqrt{\frac{hp}{\lambda S}}\right)$, we obtain:

$\phi_0 = \sqrt{h\lambda Sp}\,\theta_0$ (9a)

We can therefore introduce a thermal resistance of the infinite fin, R_{th}, such that:

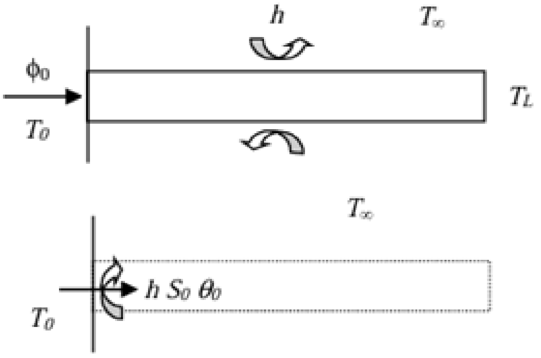
$$\phi_0 = \frac{(T_0 - T_\infty)}{R_{th}} = \frac{\theta_0}{R_{th}}$$

$$\Rightarrow R_{th} = \frac{1}{\lambda S m} = \frac{1}{\sqrt{h\lambda Sp}}$$

6-2-3. Efficiency and effectiveness of a fin

Recall that fins are used to increase the heat flux transferred from the solid to the environment. However, it should be noted that the fin itself has a thermal resistance. Thus, it may happen that if the fin is not properly sized, its presence will not contribute to an increase in heat transfer. The effectiveness of a fin is defined as the ratio between the heat flux dissipated by the fin, ϕ_0, and the heat flux that would be dissipated without the fin:

$$\varepsilon_0 = \frac{\phi_0}{hS_0\theta_0}$$



S_0 is the cross-sectional area at the base of the fin (at $x = 0$, contact with the solid)

$\theta_0 = T_0 - T_\infty$ where T_0 is the temperature at the base of the fin or at the surface of the solid.

In the case of an "infinite" fin, the effectiveness is written as:

$$\varepsilon_0 = \frac{\sqrt{h\lambda S p}\,\theta_0}{hS\theta_0} \quad \Leftrightarrow \quad \varepsilon_0 = \sqrt{\frac{\lambda p}{hS}}$$

The effectiveness of a fin is considered effective if $\varepsilon_0 \geq 1$. Thus, the effectiveness of the fin is improved by:

- The choice of a material with high thermal conductivity
- The choice of fin geometry such that $\frac{p}{S}$ is high (use of thin fins)
- The choice of a "relatively" low convective heat transfer coefficient (while still ensuring a high dissipated flux. Thus, the use of fins is more justified when the flowing fluid is a gas rather than a liquid, and when heat transfer occurs by natural convection.

Another measure of fin performance is provided by the calculation of fin efficiency. This is defined as the ratio between the heat flux dissipated by a fin, ϕ_0, and the maximum heat flux that a fin could dissipate. This maximum heat flux is achieved when the temperature difference between the fin and the surrounding fluid is maximal, i.e., when the entire fin is at the base temperature:

$$\phi_{max} = hS_{ech}^{fin}(T_0 - T_\infty) = hS_{ech}^{fin}\theta_0$$

Where S_{ech}^{fin} is the heat exchange surface between the fin and the surrounding fluid.

The efficiency of a fin is then written as:

$$\eta_0 = \frac{\phi_0}{hS_{ech}^{fin}\theta_0}$$

$$0 \leq \eta_0 \leq 1$$

In many practical applications, the analysis of the thermal behavior of a system equipped with fins becomes complex if the fins used do not have a constant cross-section. Obtaining the temperature field in the fin becomes difficult, and therefore calculating the heat flux dissipated by the fin becomes complicated.

Charts or analytical expressions for the efficiency, η_0 , and the heat exchange surface, S_{ech}^{fin}, for commonly shaped fins are then available in the literature. These allow the determination of the heat flux, ϕ_0 , given the temperature at the fin base, θ_0

Effectiveness and efficiency of a finned surface

The effectiveness of a finned surface is defined as the ratio between the total heat flux dissipated by the system with fins, ϕ_T , and the total heat flux that would be dissipated by convection without fins:

$$\varepsilon_T = \frac{\phi_T}{hS_T\theta_0}$$

Where: $S_T = NS_0 + S_{between-fins}$ is the total surface area of the system without fins in contact with the surrounding fluid, with $S_{between-fins}$ being the area between the fins. In practice, one obviously aims to design a system for which:

$$\varepsilon_T \geq 1$$

$$\phi_T = flux\ dissipated\ by\ fins + flux\ dissipated\ by\ convection\ between\ fins$$

$$\phi_T = N\phi_0 + hS_{between-fins}\theta_0$$

- ϕ_0 heat flux dissipated by one fin
- N number of fins (all identical, with base cross-sectional area, S_0) arranged on the surface S_T
- $\theta_0 = T_0 - T_\infty$, where T_0 is the temperature at the fin base or the solid surface.

$$\begin{aligned}\phi_T &= N\phi_0 + h(S_T - NS_0)\theta_0\\ &= N\eta_0 hS_0\theta_0 + h(S_T - NS_0)\theta_0\\ &= hS_T\theta_0 + Nh\theta_0 S_0(\varepsilon_0 - 1)\end{aligned}$$

- ε_0 effectiveness of one fin $\Longrightarrow \varepsilon_T = 1 + N\frac{S_0}{S_T}(\varepsilon_0 - 1)$

➢ Another measure of the performance of a finned surface is provided by calculating the efficiency of the system. This is defined as the ratio between the heat flux dissipated by the system with fins, ϕ_T, and the maximum total heat flux:

$$\eta_T = \frac{\phi_T}{hS_{ech}^{total}\theta_0}$$

Where : $S_{ech}^{total} = NS_{ech}^{fin} + S_{between-fins}$ is the total heat exchange surface area of the system with fins in contact with the surrounding fluid

$$\phi_T = N\phi_0 + hS_{between-fins}\theta_0 = N\phi_0 + h(S_{ech}^{total} - NS_{ech}^{ailett})\theta_0$$

$$= N\eta_0 hS_{ech}^{fin}\theta_0 + h\left(S_{ech}^{total} - NS_{ech}^{fin}\right)\theta_0$$

$$= hS_{ech}^{total}\theta_0 + Nh\theta_0 S_{ech}^{fin}(\eta_0 - 1)$$

$$\Longrightarrow \eta_T = 1 - N\frac{S_{ech}^{fin}}{S_{ech}^{total}}(1 - \eta_0)$$

In practice, knowing η_0 from charts, one can calculate η_T for the system under study, and then the total dissipated heat flux.

Exercises (Chap. 3)

Ex.1:

Find the expression for the temperature profile in a wall of thickness e with conductivity λ and whose face at x = 0 is subjected to a constant flux φ0 while the other face at x = e exchanges its heat by convection (with coefficient h) with a fluid at temperature T_f.

Ex .2:

Consider a concrete wall with thermal conductivity λ_b and thickness e_b, in contact on one face with a medium at temperature T_∞^{int} (for example, the interior of a living space), and the other face in contact with a medium at temperature T_∞^{ext} (the exterior). Heat exchange between the wall and its surroundings occurs by convection, due to wind outside and air movement within the room. The convective heat transfer coefficients h_{int} and h_{ext} characterize the exchanges with the room and with the exterior, respectively. Given: $T_\infty^{int} = 20°C$; $T_\infty^{ext} = -5°C$; $h_{int} = 5\frac{W}{m^2K}$; $h_{ext} = 10\frac{W}{m^2K}$; $\lambda_b = 2\frac{W}{mk}$; $e_b = 10\ cm.$

- Calculate the heat flux through the wall;
- Calculate the temperature of the wall inside the room;

If we add a layer of insulation (glass wool) with a thermal conductivity of $\lambda_{iso} = 0.04\frac{W}{mk}$ and a thickness of $e_{iso} = 10\ cm$;

- Calculate the heat flux through the wall for a heat exchange surface area S = 1 m²;
- Calculate the temperature of the wall inside the room.

Ex .3:

A double-glazed unit consists of two panes of glass, each of thickness e, separated by a layer of still air of thickness L. There is a temperature difference ΔT between the extreme surfaces of the double glazing.

1. Calculate the heat losses for a pane of glass with a surface area of 1 m², neglecting convection on either side of the glass.

2. Compare these losses to those resulting from a single pane of glass of thickness e and the same surface area for the same temperature difference ΔT.

3. Calculate the intermediate temperature differences. Comment on the results.

Glass: **$e=3.5$ mm, $\lambda_v=0.7$ W/mK ; Air : $L=12$ mm, $\lambda_a=0.024$ W/mK ; $\Delta T=5°C$**

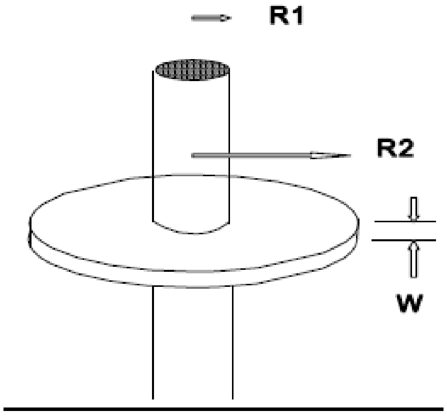
Ex 4 :

A circular fin of thickness **W** and outer radius **R_2** is installed on a tube of outer radius **R_1**, whose surface is at temperature **T_1**. The surrounding air is at temperature **T_∞**, and the convection coefficient on the fin surface is h.

a) Perform a heat balance on an appropriate volume element of the fin and obtain the differential equation that the temperature of this fin must satisfy. (Note: You are not asked to solve this equation.)

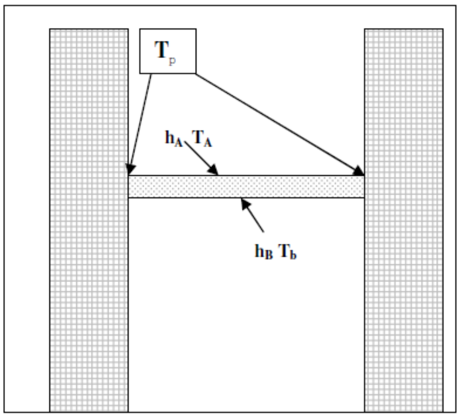
b) What are the possible different boundary conditions?

c) If the heat loss through this fin is denoted **q_c** (in watts), give the expressions for the fin efficiency and the fin effectiveness.



Ex 5 :

A fin of width **W**, thickness **e**, and length **L** is attached between two metal parts whose temperatures are identical and equal to **T_p**. The two faces of the fin are not subjected to the same conditions. The upper face of the fin is exposed to an air current at temperature **T_A** with a heat transfer coefficient **h_A**. The lower face, meanwhile, is exposed to another air current at temperature **T_B** with a heat transfer coefficient **h_B**. To determine the axial temperature profile, perform a heat balance on an appropriate volume. Clearly state your assumptions and obtain the differential equation that the temperature must satisfy.

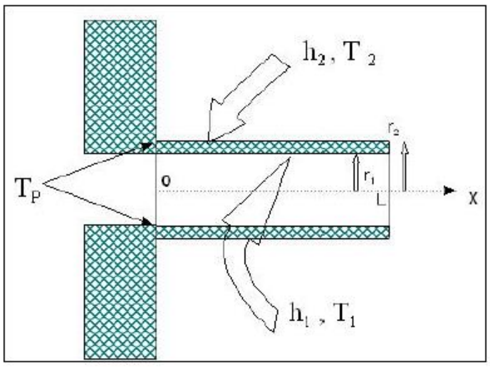


- State the boundary conditions.
- Obtain the general expression for the temperature profile as well as the relations that the integration constants in this expression must satisfy.

Ex 6 :

A fin of length **L** is in the shape of a hollow cylinder. Its base is attached to a wall at temperature **Tp**. The inner surface of the cylinder is cooled by air at temperature $\mathbf{T_1}$ with a convection coefficient $\mathbf{h_1}$, while the outer surface is cooled by air at temperature $\mathbf{T_2}$ with a convection coefficient $\mathbf{h_2}$.

It will be assumed that the axial heat flux at **x = L** at the tip of the fin is negligible.



a- Perform a heat balance on an appropriate volume element of the fin and obtain the differential equation that the temperature of this fin must satisfy.

b- What are the boundary conditions?

c- Obtain the expression for the temperature profile.

Answer : $C_2 = \frac{U_p e^{mL}}{e^{mL}+e^{-mL}}$

Heat convection

In this chapter, we consider two objectives: the mechanism of heat transfer by forced convection on the one hand, and by natural convection on the other. This study is based on dimensionless numbers, developed using dimensional analysis.

4-1. Introduction

Consider a fluid flow at a velocity V and a temperature T_∞ over an arbitrary surface, as shown in Figure 4.1. If the surface temperature T_s is different from T_∞, the convective heat transfer can be written as:

$$\varphi = h(T_s - T_\infty) \qquad (4.1)$$

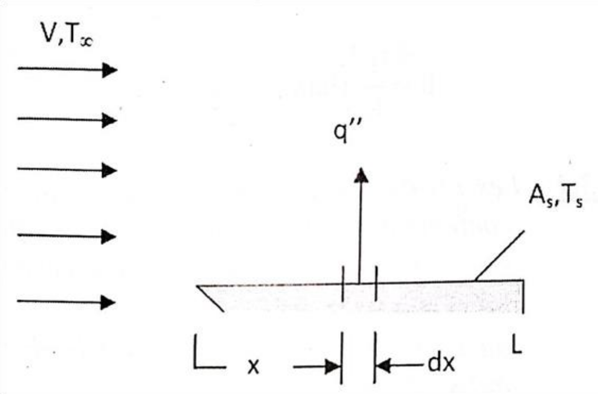


Figure 4.1: Forced convection

H being the local convection coefficient. Since h and φ vary along the surface, the total flux is written:

$$\phi = \int_S \varphi dS \qquad (4.2)$$

This gives :

$$\phi = hS(T_s - T_\infty) \qquad (4.3)$$

The expression for the average coefficient $\bar{h}$, which corresponds to the local heat transfer coefficient averaged over the entire surface of the wall in contact with the fluid, is written:

$$\bar{h} = \frac{1}{S}\int_S hdS \qquad (4.4)$$

The convective exchange coefficient, which gives access to the calculation of the heat flux exchanged between the wall and the fluid, is directly related to the thicknesses of the boundary layers and is thus an extremely difficult quantity to evaluate. This coefficient is difficult to calculate precisely but we can nevertheless give orders of magnitude in$[W.m^{-2}.K^{-1}]$:

The convective heat transfer coefficient, which allows calculation of the heat flux exchanged between the wall and the fluid, is directly related to boundary layer thicknesses and is therefore extremely difficult to evaluate. This coefficient is difficult to calculate precisely, but orders of magnitude can be given $[W.m^{-2}.K^{-1}]$:

- Forced convection : gas $h \sim 100$, liquid $h \sim 10^3$ à 10^5. Application: heat exchangers, electronic circuit coolers…
- Natural convection: gas$h \sim 10$, liquid$h \sim 10^2$. Application: building heating, meteorology….

Example 4.1

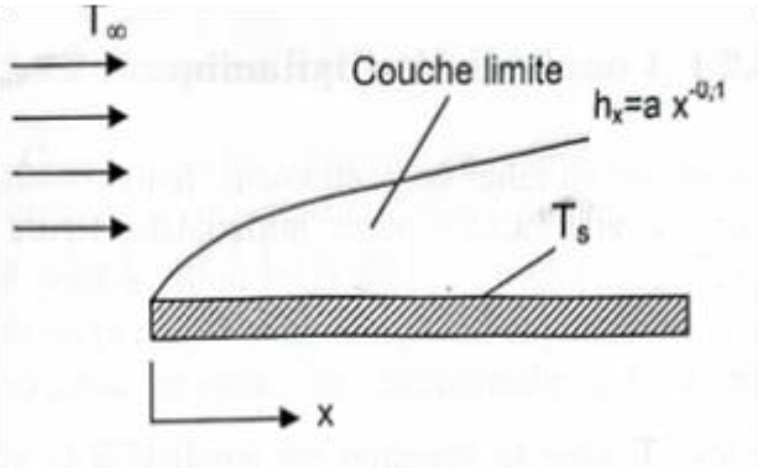
Experimental results have shown that the local convection coefficient h_x for flow over a flat surface is written :

$$h_x = ax^{-0.1}$$

Where ***a*** is a constant and ***x*** is the distance from the edge of the plane.

- Determine the ratio $\frac{\overline{h}_x}{h_x}$ at a distance x.
- Give the variation of $\overline{h_x}$ and h_x as a function of x

Solution:



- Equation (4.4) gives :

$$\bar{h} = \frac{1}{S}\int_S \ hdS = \frac{1}{x}\int_0^x h_x dx$$

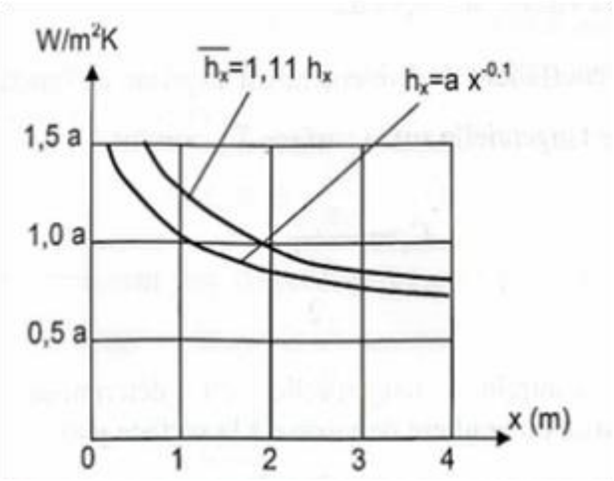
After solving this integral, we will have:

$$\overline{h_x} = 1{,}11ax^{-0{,}1} = 1{,}11h_x$$

Hence the report:

$$\frac{\bar{h}_x}{h_x} = 1{,}11$$

- The variations of h_x et $\bar{h}_x$ are represented by the figure below



4-2. Boundary layer concepts

4-2-1. Dynamic Boundary layer

To highlight this phenomenon, we consider a flow over a horizontal surface, Figure 4.2. In contact with the surface, the velocity is zero, and within the boundary layer, which is defined by its thickness $y = \delta$, the velocity depends on the thickness « y ». This phenomenon is associated with tangential stresses τ whose direction is parallel to the fluid velocity.

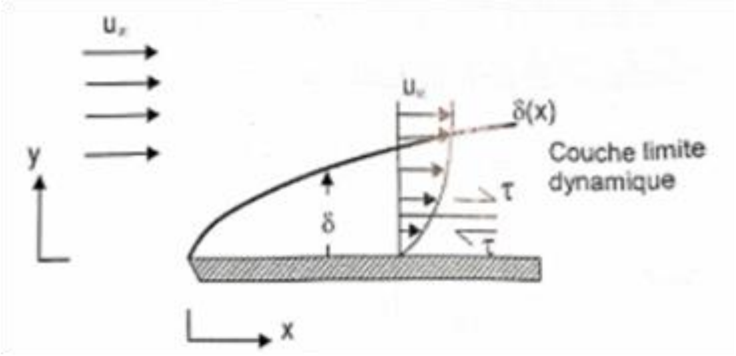


Figure 4.2 : Dynamic boundary layer on a horizontal plane

The boundary layer thickness δ is defined as the value of y when the velocity $u = 0.99\, u_\infty$

The friction coefficient is expressed as a function of the shear stress on the surface τ_s as :

$$C_f = \frac{\tau_s}{\rho \frac{u_\infty^2}{2}} \qquad (4.5)$$

The shear stress is determined by knowing the velocity gradient at the surface $y = 0$:

$$\tau_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \qquad (4.6)$$

μ being the dynamic viscosity $[N.s/m^2]$

4-2-2. Thermal boundary layer

This boundary layer develops when the surface temperature and the free flow of the fluid are different. Figure 4.3 clearly illustrates this phenomenon.

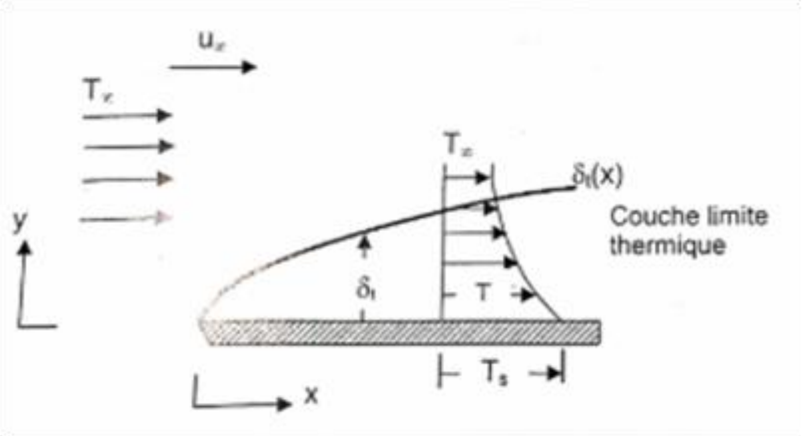


Figure 4.3. Thermal boundary layer on a horizontal plane

This thermal layer δ_t is defined as the thickness y for which the ration $\frac{T_S-T}{T_S-T_\infty}=0.99$

At any distance from the leading edge of the plate, the local heat flux is obtained by applying Fourier's law for $y=0$, as :

$$\varphi=-\lambda\left.\frac{\partial T}{\partial y}\right|_{y=0} \qquad (4.7)$$

Combining the two equations (4.1) and (4.7), we obtain:

$$h=\frac{-\lambda\left.\frac{\partial T}{\partial y}\right|_{y=0}}{T_S-T_\infty} \qquad (4.8)$$

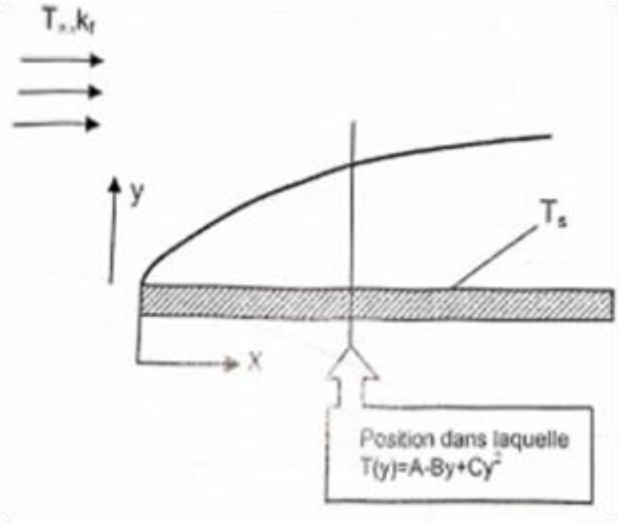
where λ is the thermal conductivity of the fluid in question.

Example 4.2

The temperature profile at distance *xis* expressed as: $T(x)=A-By+Cy^2$

A, B and C are constants.

- Determine the expression of the local convection coefficient h_x



Solution:

From equation (4.8), the coefficient is:

$$h = \frac{-\lambda(-B + 2Cy)|_{y=0}}{T_s - T_\infty}$$

What gives:

$$h = \frac{\lambda B}{T_s - T_\infty}$$

4.2.3. Laminar and turbulent flows

To solve a convection problem, it is necessary to determine the type of flow to consider (laminar or turbulent).

Figure (4.4) shows the different flow conditions between laminar and turbulent. The intermediate region is defined as transient.

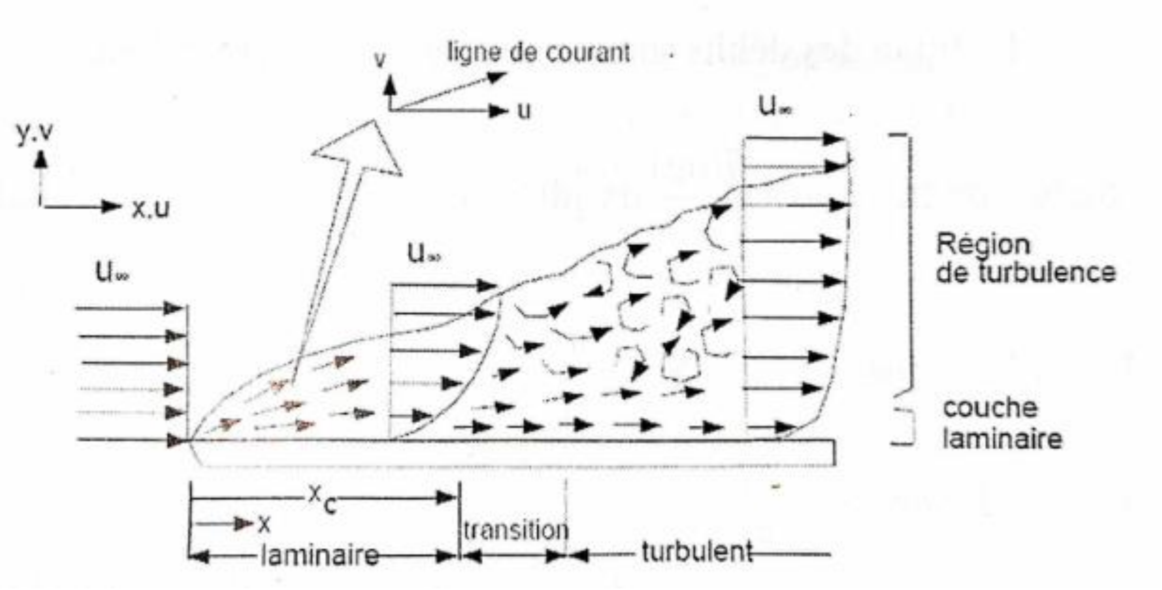


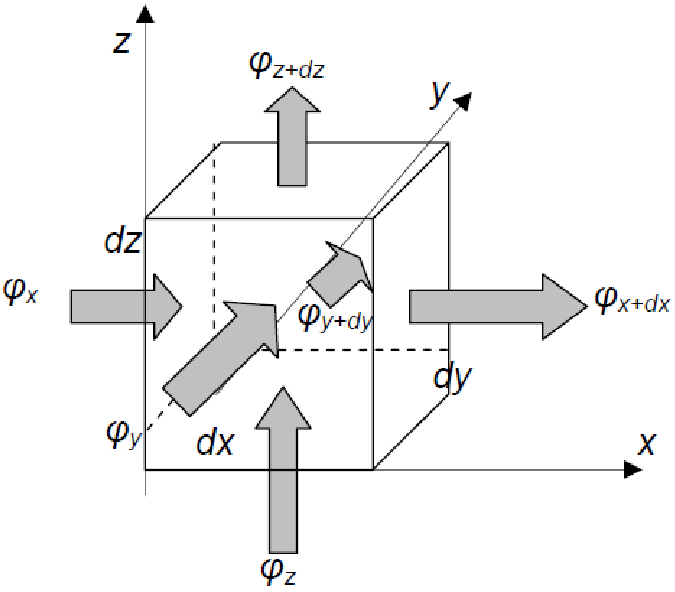
Figure 4.4: Development of the dynamic boundary layer on a flat surface.

Thex_c transient region appears from a certain so-called critical distance10^5 whose Reynolds number varies between up to 3.10^6

$$Re_{x,c} = \frac{\rho V_\infty x_c}{\mu} = 5.10^5 \qquad (4.9)$$

4.3. Heat equation:

We develop the heat equation in a fluid in motion in differential form in a Cartesian system. We consider that the flow is 3D, its velocity components along the directions x, y and z are, respectively,$u(x, y, z, t)v(x, y, z, t)$ and$w(x, y, z, t)$ We apply the first principle of thermodynamics, the principle of energy conservation, on a control volume represented by an infinitesimal parallelepiped volume element that is isolated from the flowing fluid:



The 6 facets of the volume element represent the boundary or outer surface through which thermal flow enters or exits the volume.

The energy balance allows to say that:

$$\phi_{net} + P = \Delta E \qquad (4.10)$$

Which reflects the fact that theφ_{net} net heat flow$P$ stored in the elementary volume added to the possible generation of thermal power is used to increase the internal thermal energyΔE of this elementary volume.

The heat flux, which enters or exits through a facet, is composed ofφ the conductive heat fluxφ_{cond} and the convective heat flux φ_{conv}

$$\phi = \phi_{cond} + \phi_{conv} \qquad (4.11)$$

The heat flux transmitted by conduction through the left surface of the volume element in the x *direction* is:

$$\phi_{x\ cond} = -\lambda_f dydz \left.\frac{\partial T}{\partial x}\right|_x \qquad (4.12)$$

The fluid penetrates the same facet as thermal energy:

$$\phi_{x\ conv} = \rho u dydz c_p T(x) \qquad (4.13)$$

The global thermal flux (conductive + convective) transported to the elementary volume through its right-hand surface can be obtained by developing into a Taylor series and taking only two first terms of the series as a reasonable approximation:

$$\phi_{x+dx\,cond} = \phi_{x\,cond} + \left.\frac{\partial \phi_{x\,cond}}{\partial x}\right|_x dx + \Lambda \tag{4.14}$$

$$\phi_{x+dx\,conv} = \phi_{x\,conv} + \left.\frac{\partial \phi_{x\,conv}}{\partial x}\right|_x dx + \Lambda \tag{4.15}$$

The net heat flux transmitted in the x direction is:

$$\phi_{x\,net} = \phi_x - \phi_{x+dx} = (\phi_{x\,cond} - \phi_{x+dx\,cond}) + (\phi_{x\,conv} - \phi_{x+dx\,conv}) = \left(-\frac{\partial}{\partial x}\left(-\lambda\frac{\partial T}{\partial x}\right)dxdydz\right) - \left(\rho u c_p\frac{\partial T}{\partial x}dxdydz + \rho c_p T\frac{\partial u}{\partial x}dxdydz\right) \tag{4.16}$$

In the directions y and z we also have:

$$\phi_{y\,net} = \phi_y - \phi_{y+dy} = \left(\phi - \phi_{y+dy\,cond}\right) + \left(\phi_{y\,conv} - \phi_{y+dy\,conv}\right) = \left(-\frac{\partial}{\partial y}\left(-\lambda\frac{\partial T}{\partial y}\right)dxdydz\right) - \left(\rho v c_p\frac{\partial T}{\partial y}dxdydz + \rho c_p T\frac{\partial v}{\partial y}dxdydz\right) \tag{4.17}$$

Then:

$$\phi_{z\,net} = \phi_z - \phi_{z+dz} = (\phi_{z\,cond} - \phi_{z+dz\,cond}) + (\phi_{z\,conv} - \phi) = \left(-\frac{\partial}{\partial z}\left(-\lambda\frac{\partial T}{\partial z}\right)dxdydz\right) - \left(\rho w c_p\frac{\partial T}{\partial z}dxdydz + \rho c_p T\frac{\partial w}{\partial z}dxdydz\right) \tag{4.18}$$

The net thermal flux transmitted in all directions or exchanged by the fluid element through its outer surface with the external medium:

$$\phi_{net} = \phi_{net\,cond} + \phi_{net\,conv} = \left[\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)\right]dxdydz - \left(\rho u c_p\frac{\partial T}{\partial x}dxdydz + \rho v c_p\frac{\partial T}{\partial y}dxdydz + \rho w c_p\frac{\partial T}{\partial z}dxdydz\right) - \rho c_p T\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dxdydz \tag{4.19}$$

It may be that within the volume element there is heat generation by a volumetric power$P[\frac{W}{m^3}]$ released, for example, following a chemical or nuclear reaction, or electrical effects. The net heat flow that enters or leaves the volume element added to the thermal power, together, serve to change the internal thermal energy stored in the control volume:

$$\rho c_p\frac{\partial T}{\partial t}dxdydz \tag{4.20}$$

Where c_p and ρ are the heat and density of the medium.

The continuity equation allows to write

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (4.21)$$

Finally, the principle of energy conservation gives us the general heat equation in a fluid:

$$\rho c_p\frac{\partial T}{\partial t}+\rho c_p u\frac{\partial T}{\partial x}+\rho c_p v\frac{\partial T}{\partial y}+\rho c_p w\frac{\partial T}{\partial z}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)+p_v \qquad (4.22)$$

It is assumed that the medium is homogeneous and isotropic, so λ, ρ and c are constant regardless *of* x, y and z. Also, we will only deal with stationary problems, 2D and without heat generation.

Solving the problem of thermal convection, in other words, determining the convection coefficient and subsequently the heat flow exchanged between a moving fluid and a wall amounts to solving a system of four differential equations to calculate the four variables, the speed field, $u(x,y)$, $v(x,y)$ and $w(x,y)$ and the pressure $p(x,y)$ and then the temperature field $T(x,y)$.

The system of differential equations consists of:

The momentum equations, following x and following y:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad (4.23)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \qquad (4.24)$$

The equation of continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (4.25)$$

And the heat equation:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \qquad (4.26)$$

Where $\alpha=\frac{\lambda_f}{\rho c_p}$ is the thermal diffusivity of the fluid.

Classification

Depending on the source of the fluid’s motion, convection is classified into two types:

- ***forced convection***
- ***natural convection***

In forced convection, the movement of the fluid is due to an external source, pump, fan, compressor,... while in free convection the movement of the fluid is caused by the bulk density gradient due to the temperature gradient in the fluid.

For the two types of convection, we distinguish convection in confined flow (convection in an enclosed space) and convection in external flow. This is true in both laminar and turbulent flow regimes.

4.4. Correlation laws for the convective exchange coefficient:

In the field of engineering, the exchange coefficients are calculated from correlation laws, obtained either by the precise analysis of the mechanisms that govern transfers in the boundary layers, or obtained from experiments (empirical laws).

4.4.1. Characteristic parameters of convection

Heat exchanges by convection occur within a flowing fluid. We saw that the exchange coefficient, which is used in the calculation of heat flux, was related to many parameters, notably the flow regime (laminar or turbulent) and the nature of the fluid. We therefore seek to express this exchange coefficient as a function of characteristic quantities of the flow regime and the nature of the fluid. Thus we can write that h is a function of the flow velocity, the momentum and heat diffusion coefficients (which control the boundary layer thicknesses) and a characteristic length of the system.

In practice, we rather use sizes without dimensions.

- The flow regime in forced convection is characterized from a dimensionless number: the Reynolds number, which quantifies the importance of inertia forces (flow motor) compared to viscous forces (dissipation, flow brake). It is written:

$$Re_{L_{ref}} = \frac{forces\ d'inertie}{forces\ visqueuses} = \frac{\frac{\rho U^2}{L_{ref}}}{\mu \frac{U}{L_{ref}^2}} = \frac{U L_{ref}}{\nu} \qquad (4.27)$$

Where:

U is the characteristic speed of the flow $(m.s^{-1})$

L_{ref} is a characteristic length of the studied system (m)

ν is the fluid’s kinematic viscosity (or momentum diffusivity), defined by:

$$\nu = \frac{\mu}{\rho}$$

Where

μ is the dynamic viscosity of the fluid $(kg.m^{-1}.s^{-1})$

ρ is the density of the fluid $(kg.m^{3})$

ν is thus a physical property of the fluid, which quantifies the ability of this fluid to diffuse momentum (to attenuate velocity gradients). She expresses herself in $(m^2 s^{-1})$

- The flow regime in natural convection is characterized from a dimensionless number: the Grashof number, which quantifies the importance of Archimedean forces (flow motor) compared to viscous forces (dissipation, flow brake). It is written:

$$Gr_{L_{ref}} = \frac{\rho\beta\Delta T L_{ref}^3}{\nu^2} \qquad (4.28)$$

Where:

g is the acceleration of gravity $(m.s^{-2})$

β is the coefficient of thermal expansion(K^{-1}) : $\beta = \frac{1}{\rho}\frac{\partial\rho}{\partial T}\Big)_p$

ΔT is a characteristic temperature difference of the studied system (K)

The behavior of the fluid with respect to convective heat exchanges is characterized by the Prandtl number. It is a dimensionless parameter defined by the ratio between momentum diffusivity and thermal diffusivity:

$$Pr = \frac{\nu}{\alpha} \qquad (4.29)$$

α is the thermal diffusivity of the fluid, defined by: $\alpha = \frac{\lambda}{\rho c_p}$

Whereλ is the thermal conductivity of the fluid $(W.m^{-1}.K^{-1})$

c_p is the specific heat of the fluid $(J.kg^{-1}.K^{-1})$

α is thus a property of the fluid, which quantifies the ability of this fluid to diffuse heat. She expresses herself in m^2/s.

The Prandtl number can be seen as a ratio of two characteristic times: the momentum diffusion time and the thermal diffusion time.

- Prandtl number can be combined with the Reynolds number to form the Péclet number

$$Pe_{L_{ref}} = Re_{L_{ref}} Pr = \frac{UL_{ref}}{\alpha} \qquad (4.30)$$

- Prandtl number can be combined with Grashof number to form Rayleigh number

$$Ra_{L_{ref}} = Gr_{L_{ref}} Pr = \frac{g\beta\Delta T {L^3}_{ref}}{\nu\alpha} \qquad (4.31)$$

The heat flux exchanged by convection will be characterized by comparing it to a reference heat flux exchanged by conduction. We thus define a dimensionless number, called the Nusselt number:

$$Nu_{ref} = \frac{flux\ convectif}{flux\ de\ conduction\ de\ référence} = \frac{loi\ de\ Newton}{loi\ de\ Fourier} = \frac{hS\Delta T}{\lambda S \frac{\Delta T}{L_{ref}}}$$

$$Nu_{ref} = \frac{hL_{ref}}{\lambda} \qquad (4.32)$$

A high Nusselt number will therefore mean that convective heat exchanges predominate over conductive exchanges.

Note:

Local Nusselt number is defined from the local exchange coefficient associated with the heat flow exchanged locally between a wall and the fluid:

$$Nu_x = \frac{h(x).x}{\lambda}$$

Number of average Nusselt is defined from the average exchange coefficient associated with the overall heat flux over the entire surface of the wall:

$$Nu_L = \frac{\bar{h}.L}{\lambda} \qquad \text{where} \qquad \bar{h} = \frac{1}{L}\int_0^L h_x dx$$

Reynolds number characteristic of the flow at a position x along the plate $(L_{ref} = x)$

$$Re_x = \frac{U.x}{\nu}$$

4.4.2. Correlation laws in convection

Convective heat transfer depends on the flow regime (laminar or turbulent) and the nature of the fluid.

In forced convection, we will therefore try to establish correlations that link the Nusselt number to the Reynolds and Prandtl numbers:

$$Nu_{L_{ref}} = f(Re_{L_{ref}}, Pr)$$

In natural convection, we will try to establish correlations that link the Nusselt number to the Grashof and Prandtl numbers. Studies show that correlations are simply written using the Rayleigh number:

$$Nu_{L_{ref}} = f(Ra_{L_{ref}})$$

4.4.3. Methodology for calculating convection heat flux

- Calculation of the Reynolds number,$Re_{L_{ref}}$ (forced convection) or the Rayleigh number (natural convection), and the Prandtl number Pr$Ra_{L_{ref}}$
- Choice of the correlation
- Nusselt number calculation:

$Nu_{L_{ref}} = f(Re_{L_{ref}}, Pr)$ or $Nu_{L_{ref}} = f(Ra_{L_{ref}})$

- Calculation of the exchange coefficient (local or average): $h = \frac{\lambda}{L_{ref}} Nu_{L_{ref}}$
- Calculation of the heat flux (local or global) by Newton's law.

Note:

In the case of pipes, the characteristic length is the hydraulic diameter defined by:

$$D_h = 4\frac{Section\ de\ passage\ du\ fluide}{Périmètre\ mouillé\ de\ la\ conduite} = 4\frac{S}{p}$$

Examples:

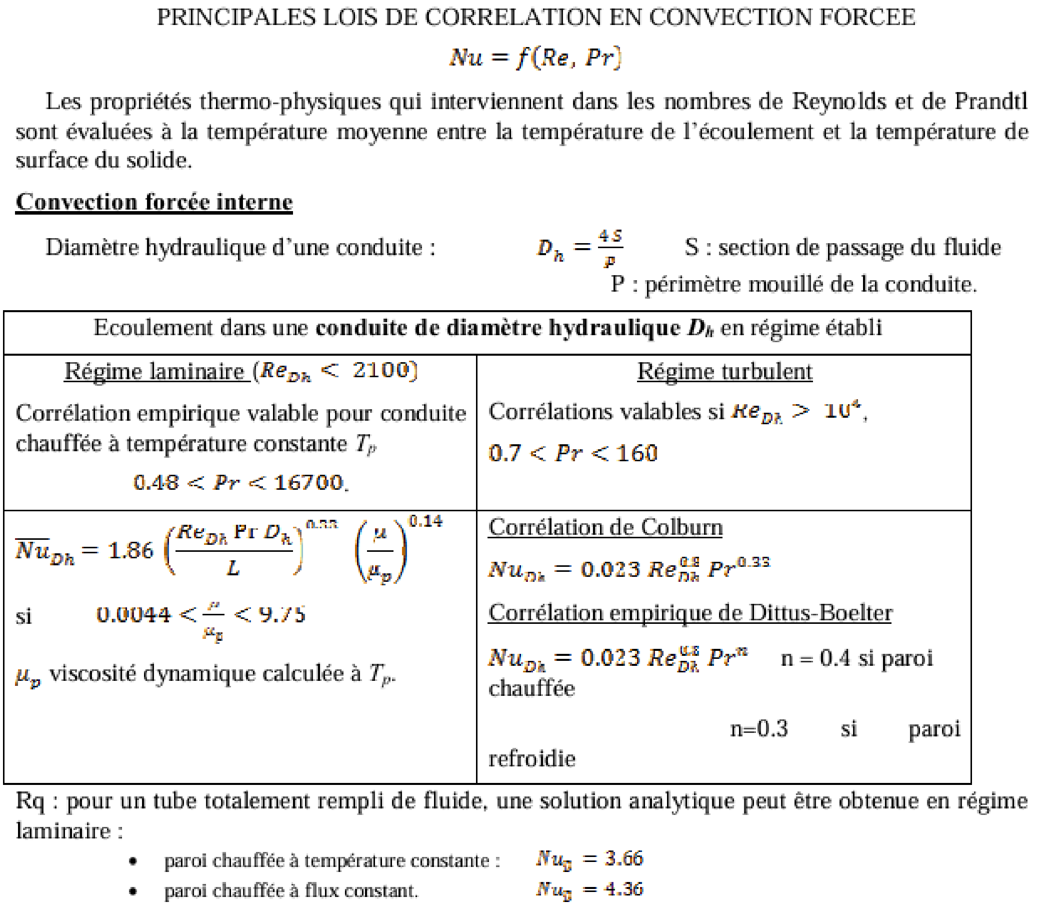
- Cylinder of diameter D completely filled with fluid:

$S = \pi \frac{D^2}{4}$ and $p = \pi D$ $D_h = D$

- Rectangular pipe of height H, width L completely filled with fluid:

$S = HL$ $p = 2(H + L)$ $D_h = 2\frac{HL}{H+L}$

In the case where $L \gg H$ $D_h = 2\frac{HL}{H+L} \approx 2\frac{H}{\frac{H}{L}+1} \approx 2H$



PRINCIPALES LOIS DE CORRELATION EN CONVECTION FORCEE

$$Nu = f(Re, Pr)$$

Les propriétés thermo-physiques qui interviennent dans les nombres de Reynolds et de Prandtl sont évaluées à la température moyenne entre la température de l'écoulement et la température de surface du solide.

Convection forcée interne

Diamètre hydraulique d'une conduite : $D_h = \frac{4S}{P}$ S : section de passage du fluide
P : périmètre mouillé de la conduite.

Ecoulement dans une **conduite de diamètre hydraulique D_h** en régime établi	
Régime laminaire ($Re_{Dh} < 2100$) Corrélation empirique valable pour conduite chauffée à température constante T_p $0.48 < Pr < 16700$.	Régime turbulent Corrélations valables si $Re_{Dh} > 10^4$, $0.7 < Pr < 160$
$\overline{Nu}_{Dh} = 1.86 \left(\frac{Re_{Dh}\, Pr\, D_h}{L}\right)^{0.33} \left(\frac{\mu}{\mu_p}\right)^{0.14}$ si $0.0044 < \frac{\mu}{\mu_p} < 9.75$ μ_p viscosité dynamique calculée à T_p.	Corrélation de Colburn $Nu_{Dh} = 0.023\, Re_{Dh}^{0.8}\, Pr^{0.33}$ Corrélation empirique de Dittus-Boelter $Nu_{Dh} = 0.023\, Re_{Dh}^{0.8}\, Pr^{n}$ n = 0.4 si paroi chauffée n=0.3 si paroi refroidie

Rq : pour un tube totalement rempli de fluide, une solution analytique peut être obtenue en régime laminaire :

- paroi chauffée à température constante : $Nu_D = 3.66$
- paroi chauffée à flux constant. $Nu_D = 4.36$

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<table>
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Corrélation empirique valable pour conduite chauffée à température constante T_p
$0.48 < Pr < 16700$.</td><td>Régime turbulent
Corrélations valables si $Re_{Dh} > 10^4$,
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Rq : pour un tube totalement rempli de fluide, une solution analytique peut être obtenue en régime laminaire :

- paroi chauffée à température constante : $Nu_D = 3.66$
- paroi chauffée à flux constant. $Nu_D = 4.36$

Convection forcée externe

Ecoulement autour d'un cylindre de diamètre D : corrélation de Hilpert $Pr \geq 0.7$
$\overline{Nu}_D = C\, Re_D^{m}\, Pr^{0.33}$

Re_D	C	m
0.4 – 4	0.989	0.33
4 – 40	0.911	0.385
40 – 4 10^3	0.683	0.466
4 10^3 - 4 10^4	0.193	0.618
4 10^4 - 4 10^5	0.027	0.805

Ecoulement autour d'une **sphère de diamètre *D* :** corrélation de Whitaker valable pour : $3.5 \leq Re_{Dh} \leq 8\ 10^4$ et $0.7 < Pr < 380$
$\overline{Nu}_D = 2 + \left(0.4\, Re_D^{0.5} + 0.06\, Re_D^{0.66}\right) Pr^{0.4} \left(\frac{\mu_\infty}{\mu_S}\right)$

μ_∞ est la viscosité dynamique du fluide calculée à la température de l'écoulement à l'infini (loin de la sphère).

μ_S est la viscosité dynamique du fluide calculée à la température de surface de la sphère.

Ecoulement le long d'une **plaque plane de longueur *L***	
Régime laminaire	Régime turbulent
Local : $Nu_x = 0.332\, Re_x^{0.5}\, Pr^{0.33}$ $(Pr \geq 0.6)$ Moyen : $\overline{Nu}_L = 0.664\, Re_L^{0.5}\, Pr^{0.33}$	$Nu_x = 0.0296\, Re_x^{0.8}\, Pr^{0.33}$ $(0.6 < Pr < 60)$ $\overline{Nu}_L = 0.037\, Re_L^{0.8}\, Pr^{0.33}$

4.5. Dimensional analysis

It is obvious that the use of equations to deal with convective heat transfer problems encounters many problems, particularly in industrial applications where installations and their operation are generally very complex.

However, experimental data are often available (heat flux, temperatures, pressure drops...). The problem is then to relate the measured quantities to the characteristic quantities of the considered flow. One can then use a dimensional analysis method.

This method first consists of writing the searched relationship in the form of a product of these characteristic quantities each affected by an a priori unknown exponent. Each quantity, including the tested quantity, is then written in dimensional form (product of fundamental dimensions, mass M, length L, temperature θ, time t, ...).

The exponents must then check linear relations (four if we stick to dimensions M, L, θ, t) so that the relation is homogeneous. We then deduce the functional dependence of the quantity analyzed in dimensionless form.

4.5.1. Dimensional analysis.

1- Remarks

The complexity of convection phenomena makes it necessary to use general techniques that allow limiting the number of parameters of a problem. These techniques are dimensional analysis and similarity.

The common goal of dimensional analysis and similarity is to determine, from the intervening physical dimensional quantities, a smaller number of dimensionless groupings.

To apply the methods of dimensional analysis, it is enough to know all the physical quantities involved, as well as their dimension. On the other hand, similarity methods require knowledge of the equations governing the phenomenon

To describe a physical phenomenon amounts, in a general way, to establishing one or more relationships linking physical quantities that describe the physical phenomenon concerned.

Dimensional analysis differs from other methods of approach in that it does not give equations that can be solved, and what limits it is the fact that it does not provide information about the nature of the phenomenon. Also, to apply dimensional analysis, it is essential to first know the variables that influence the phenomenon, which requires experience.

2- Principle:

Dimensional analysis requires simple mathematical calculations; its scope is the widest.

To apply dimensional analysis, it is essential to first know the variables that influence the phenomenon, and the success or failure of the method depends on the appropriate choice of these variables.

Every phenomenon is a function of a certain number of quantities (E_1,E_2,E_3, ,E_p) independent or related.

Establishing a physical law consists of finding a mathematical relationship between the different variables E_1, E_2, ..., E_p involved in the phenomenon.

If the relationship exists, it can be expressed in the form:

$$f(E_1, E_2, E_3, ..., E_p) = 0$$

This is the most general form of any complete physical equation at the equilibrium of the system.

The phenomenon is independent of the units chosen for the different quantities.

The same must be true for the relationship ***f***.

Applying dimensional analysis allows us to transform

$$f(E_1, E_2, E_3,...,E_p) = 0 \qquad \text{to} \qquad F(\pi_1, \pi_2, \pi_{3,},,...) = 0$$

E_1, E_2, E_3...E_p are dimensional variables.

π_1, π_2, π_3... are dimensionless numbers or groups

To establish the expression for the dimensionless numbers appearing in F, we can use one of the following two methods:

- **The application of Buckingham's π theorem**.
- **The normalization of the differential equations describing the physical phenomenon.**

Both methods obviously lead to the same results. The second method requires knowledge of the equations of the physical phenomenon, while the first only requires knowledge of all the parameters that influence the physical phenomenon.

- **Vaschy-Buckingham's π theorem (1890-1915)**

Vaschy-Buckingham's theorem allows us to predict that the most general form of the physical law describing the phenomenon under study will be written:

$$F(\pi_1, \pi_2, \pi_3, \ldots, \pi_{p-q}) = 0$$

Where, the π_i are dimensionless groupings.

According to this theorem, the number of independent dimensionless groups that can be formed by combining the physical variables E_1, E_2...E_p of the given problem is equal to the total number of these physical quantities p minus the number of fundamental dimensions q needed to express the dimensional formulas of the p physical quantities.

If π_1, π_2..., are the dimensionless groupings, then their number is given by: Number = p – q groupings.

p: total number of physical quantities.

q: number of fundamental dimensions.

The equation that expresses the relationship between the different dimensionless groupings is given by:

$$F(\pi_1, \pi_2, \pi_3, \ldots, \pi_{p-q}) = 0$$

The first step: consists of choosing a system of fundamental dimensions. This choice is arbitrary, but the dimensional formulas of all appropriate variables E_1, E_2, ... must be expressed in terms of the fundamental dimensions in the International System of Units (SI). These are:

Length L

Time t

Temperature θ

Mass M

The dimensional formula of a physical quantity is derived from its definitions and physical laws.

Example: The length of a bar $[L]$

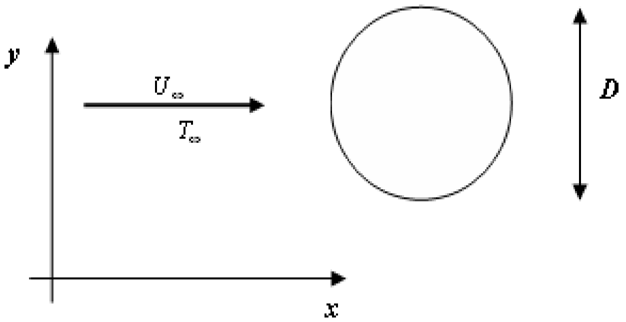
The velocity of a fluid particle $\frac{[\mathrm{L}]}{[\mathrm{t}]} = [\mathrm{Lt}^{-1}]$

The second step: determining dimensionless groups.

We propose to determine the dimensionless groups in the following cases:

Case 1: Forced convection.

Study of heat exchange by forced convection between an infinitely long cylinder heated by a fluid arriving perpendicularly to its axis and at a constant velocity:



Flow around a cylinder

Based on the description of the convective heat transfer process, it is reasonable to assume that the physical quantities considered in the table below are appropriate for the problem at hand:

Physical Quantities	**Symbol**	**Units *SI***	**Dimensional Equations**
Tube Diameter	D	m	[L]
Fluid Velocity	*U*	m/s	$[Lt^{-1}]$
Fluid Density	ρ	Kg/m^3	$[ML^{-3}]$
Dynamic Viscosity of the Fluid	μ	Kg/(m.s)	$[ML\text{-}1t^{-1}]$
Thermal Conductance of the Fluid	λ	W/(m°c)	$[MLt^{-3}\theta^{-1}]$
Heat Capacity of the Fluid at Constant Pressure.	C_P	J/(Kg°c)	$[L2t^{-2}\theta^{-1}]$
Heat exchange coefficient	h	$W/(m^2°c)$	$[Mt^{-3}\theta^{-1}]$
Difference between extreme temperatures	ΔT	°c	[θ]

That is, the physical relationship to be sought between the different variables $D, U_\infty, \rho, \mu, \lambda, C_p, h, \Delta T$ involved in the phenomenon is:

$$f\left(D, U_\infty, \rho, \mu, \lambda, C_p, h, \Delta T\right) = 0$$

Therefore, p = 8 physical quantities

q = 4 fundamental dimensions L, M, t, θ

We conclude, therefore, that we will have 8-4=4 dimensionless groupings to seek.

$$F(\pi_1, \pi_2, \pi_3, \pi_4) = 0$$

To find these groupings, we write π as a product of variables, each with an unknown exponent.

$$\pi = D^a.\lambda^b.U_\infty{}^c.\rho^d.\mu^e.C_p{}^f.h^g.\Delta T^i$$

This corresponds to the following dimensional equation for π:

$$[\pi]$$
$$= [L]^a.[M.L.t^{-3}.\theta^{-1}]^b[L.t^{-1}]^c[M.L^{-3}]^d[M.L^{-1}.t^{-1}]^e[M^2.t^{-2}.\theta^{-1}]^f[Mt^{-3}.\theta^{-1}]^g[\theta]^i$$

Or again

$$[\pi] = [M]^{b+d+e+g}.[L.]^{a+b+c-3d-e+2f}[t]^{-3b-c-e-2f-3g}[\theta]^{-b-f-g+i}$$

The grouping π must be dimensionless, that is to say

$$\begin{cases} b+d+e+g=0 \\ a+b+c-3d-e+2f=0 \\ 3b-c-e-2f-3g=0 \\ b-f-g+i=0 \end{cases}$$

This is a system of 4 equations with 8 unknowns a, b, c, d, e, f, g, i. From a mathematical point of view, 4 of the 8 unknowns can be chosen arbitrarily; the only restriction concerning this choice is that each chosen exponent must be independent of the others.

An exponent is independent if the determinant formed by the coefficients of the remaining terms is not zero. With a bit of luck and a good grasp of physics, we can solve this system by adopting the following approach:

- Since we need to determine the expression for the convective heat transfer coefficient h, it is convenient to set its exponent g = 1, and at the same time, we assume that c = d = i = 0.

By solving the system of equations above using this approach, we obtain:

$a = 1, \qquad b = -1, \qquad e = f = 0$

Therefore

$$\pi_1 = \frac{D.h}{\lambda} = Nu$$

This is called the Nusselt number.

It is the first dimensionless grouping with the choice g = 1.

We choose g = 0 (the variable h is part of a dimensionless number, which justifies this choice), and we assume that a = 1 and f = i = 0.

Solving the system, we obtain:

$b = 0, \qquad c = d = 1, \qquad e = -1$

Let us have:

$\pi_2 = \frac{\rho U_\infty D}{\mu} = Re$, which is the Reynolds number.

We choose $c = g = e = i = 0$; solving the system of equations above, we obtain:

$a = d = 0$ and $f = 1$

Let us have:

$\pi_3 = \frac{\mu C_p}{\lambda} = Pr$, which is the Prandtl number.

We choose $a = b = d = e = 0$; Solving the system of equations above, we obtain:

$g = 0 \quad c = 2 \quad i = 1$ and $f = -1$

Therefore

$\pi_4 = \frac{U_\infty^2}{C_p \Delta T} = Ec$, which is Eckert's number.

The equation giving the dimensionless groupings can then be written as:

$$F(Nu, Re, Pr, Ec)$$

Or, alternatively,

$$Nu = f(Re, Pr, Ec)$$

Eckert's number only appears in the description of flows close to the speed of sound.

The importance of this reduction of variables becomes apparent when seeking a correlation between experimental data.

Case 2: Natural (Free) Convection

Study of heat exchange; by natural convection around a uniformly heated horizontal cylinder.

Consider a fluid at rest at temperature T_0 in which a horizontal cylinder heated uniformly to temperature T_p is placed.

Assumptions:

Physical properties: λ, μ, C are assumed to be uniform.

It is assumed that $\rho = Cst = \rho_0$ is uniform, except in the terms expressing the gravitational forces (Boussinesq approximation).

$$\rho = \rho_0(1 + \beta(T - T_0))$$

$\beta = \frac{1}{V}(\frac{\partial V}{\partial T})_p = -\frac{1}{\rho}(\frac{\partial \rho}{\partial T})_p$ (coefficient of volumetric expansion at constant pressure).

In free convection, it is the differences in density within the fluid that cause its movement. A fluid particle of density ρ and volume dv located near a hot wall will be subjected to the following forces:

- Weight $\rho g dv$
- Archimedes' buoyant force $\rho_0 g dv$

The resultant of the forces acting vertically on the particle is given by:

$F = (\rho_0 - \rho) g dv$ or $f = \frac{\rho_0 - \rho}{\rho} g$ Force per unit mass

According to Boussinesq's hypothesis, $\rho_0 - \rho \ll \rho_0$ $\Rightarrow$ $f = \frac{\rho_0 - \rho}{\rho_0} g$ that is $\rho_0 - \rho$

Using the density relationship, we have:

$\frac{\rho_0 - \rho}{\rho_0} = \beta(T - T_0)$ and $f = \beta(T - T_0)$

According to the relationship established above, there is coupling between the velocity and temperature fields via the term $\beta(T - T_0)$. Finally, for the study of heat exchange by natural convection, the appropriate physical variables are:

$$(l, \nu = \frac{\mu}{\rho}, \beta g, T - T_0, \alpha, \lambda, \varphi)$$

Symbols	Quantities	SI Units	Dimensional Equations
ℓ	Length	m	[L]
ν	Kinematic viscosity	m^2/s	$[L^2 t^{-1}]$
β	Coefficient of thermal expansion	$1/°c$	$[\theta^{-1}]$
g	Acceleration	m/s^2	$[Lt^{-2}]$
T-T_O	Temperature difference = ΔT_O	°C	$[\theta]$
$\alpha = {}^{\lambda}/_{\rho C_p}$	Thermal diffusivity	m^2/s	$[L^2 t^{-1}]$
λ	Thermal conductivity of the fluid	w/m°c	$[MLt^{-3}\theta^{-1}]$
φ	Heat flux $\varphi = h(T - T_0)$	w/m^2	$[Mt^{-3}]$

That is,

$$f(l, \nu, \beta g, T - T_0, \alpha, \lambda, \varphi) = 0$$

Therefore, there are, $p = 7$ appropriate physical variables and $q = 4$, fundamental dimensions $\Longrightarrow$ $7 - 4 = 3$ dimensionless groups.

We then have:

$$F(\pi_1, \pi_2, \pi_3) = 0$$

To express these groupings, we follow the same approach as before:

$$\pi = l^a . \nu^b . (g\beta)^c . (T - T_0)^d . \alpha^e . \lambda^f . \varphi^g$$

With the following two-dimensional equation:

$$[\pi] = [L]^a.[L^2t^{-1}]^b.[Lt^{-2}\Theta^{-1}]^c.[\Theta]^d.[L^2t^{-1}]^e.[MLt^{-3}\Theta^{-1}]^f.[Mt^{-3}]^g$$

We express that the number π is dimensionless $\Longrightarrow$ a system of equations to solve, which leads to the following results:

$\pi_1 = \frac{\varphi l}{\lambda \Delta T_0} = Nu$ Nusselt number

$\pi_2 = \frac{\nu}{\alpha} = \frac{\mu}{\rho_0}\frac{\rho_0 C_p}{\lambda} = \frac{\mu C_p}{\lambda} = Pr$ Prandtl number

Then a new number that characterizes free convection

$\pi_3 = \frac{\beta g \Delta T_0 l^3 \nu}{\nu^2} = Gr$ Grashof number

The Gr number is the ratio of Archimedes' forces to viscous forces. The larger Gr is, the greater the Archimedes' forces, which influences the fluid's movement.

The equation $F(\pi_1, \pi_2, \pi_3) = 0$ can be written as:

$$F(Nu, Pr, Gr) = 0$$

Or

$$Nu = f(Gr, Pr)$$

This relationship is a correlation to be determined experimentally.

Note: The Reynolds number does not appear in natural convection, since there is no imposed flow $(U_0 = 0)$.. However, a local Reynolds number Re can be calculated, which corresponds to the motion induced by the density gradients ρ.

Limitation of the π theorem:

The π theorem is no longer applicable when the equations (system of power equations) involving the powers of the variables form a linearly dependent system; that is, if the equations are not independent, the number of dimensionless groups is equal to the total number of p variables minus the number of independent equations.

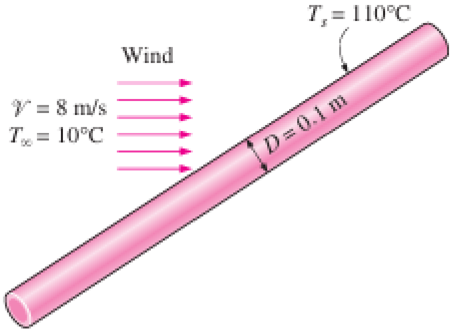
Conclusions:

Dimensional analysis is a powerful method because, in the absence of a conservation equation, it allows us to determine dimensionless groups that characterize a physical phenomenon. However, it has limitations:

- It does not provide information about the physical nature of the phenomenon.
- It only leads to significant results if the count of quantities is accurate and complete.
- It must be complemented by an experimental study to clarify the form of the relationship linking the dimensionless groupings.

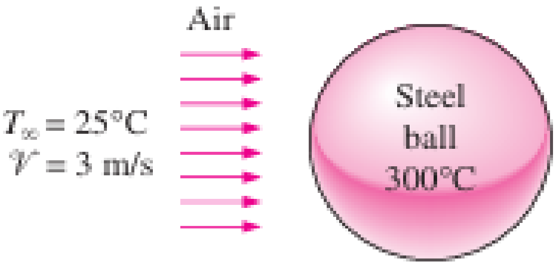
Ex 1:

A long 10-cm-diameter steam pipe whose external surface temperature is 110°C passes through some open area that is not protected against the winds (Fig). Determine the rate of heat loss from the pipe per unit of its length when the air is at 1 atm pressure and 10°C and the wind is blowing across the pipe at a velocity of 8 m/s.



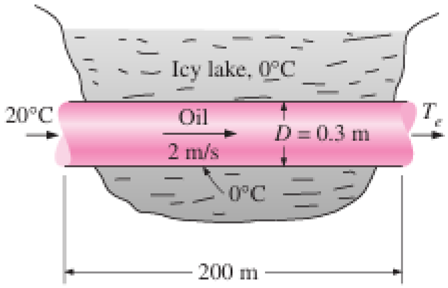
Ex 2:

A 25-cm-diameter stainless steel ball (8055 kg/m^3, Cp 480 J/kg °C) is removed from the oven at a uniform temperature of 300°C (Fig). The ball is then subjected to the flow of air at 1 atm pressure and 25°C with a velocity of 3m/s. The surface temperature of the ball eventually drops to 200°C. Determine the average convection heat transfer coefficient during this cooling process and estimate how long the process will take.



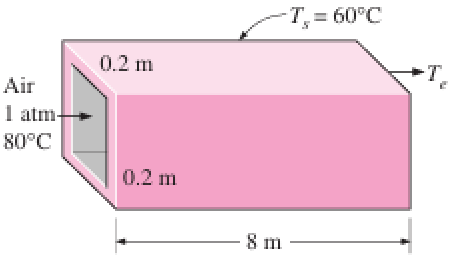
Ex 3:

Consider the flow of oil at 20°C in a 30-cm-diameter pipeline at an average velocity of 2 m/s (Fig.). A 200-m-long section of the pipeline passes through icy waters of a lake at 0°C. Measurements indicate that the surface temperature of the pipe is very nearly 0°C. Disregarding the thermal resistance of the pipe material, determine (a) the temperature of the oil when the pipe leaves the lake, (b) the rate of heat transfer from the oil, and (c) the pumping power required to overcome the pressure losses and to maintain the flow of the oil in the pipe



Ex 4:

Hot air at atmospheric pressure and 80°C enters an 8–m-long uninsulated square duct of cross section 0.2m×0.2m that passes through the attic of a house at a rate of 0.15 m^3/s (Fig). The duct is observed to be nearly isothermal at 60°C. Determine the exit temperature of the air and the rate of heat loss from the duct to the attic space



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<table>
<tr><th>SEMESTRE</th><th colspan="2">Intitulé de la matière</th><th>Coefficient</th><th>Crédits</th><th>Code</th></tr>
<tr><td>S4</td><td colspan="2">Transfert de chaleur 1</td><td>3</td><td>6</td><td>IST 4.2</td></tr>
<tr><td>VHH</td><td>Cours</td><td>Travaux dirigés</td><td colspan="3">Travaux Pratiques</td></tr>
<tr><td>45h00</td><td>1h30</td><td>1h30</td><td colspan="3">-</td></tr>
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Objectifs de l'enseignement:
Apprécier les pouvoirs conducteurs de la chaleur des matériaux usuels, évaluer les taux de transfert de chaleur par conduction en régime stationnaire pour des géométries courantes. Applications aux ailettes rectangulaires. Connaître les mécanismes des transferts de chaleur entre un fluide et une surface solide.

Connaissances préalables recommandées:
Thermodynamique, MDF, Mathématique

Contenu de la matière:

Chapitre 1. Introduction des transferts thermiques et position vis-à-vis de la thermodynamique. (1 Semaine)

Chapitre 2. Lois de base des transferts de chaleur (1 Semaines)

Chapitre 3. Conduction de la chaleur (5 Semaines)
Loi de Fourier. Conductivité thermique et ordres de grandeur pour les matériaux usuels. Discussion des paramètres dont dépend la conductivité thermique. Equation de l'énergie, les hypothèses simplificatrices et les différentes formes. Les conditions aux limites spatiales et initiales. Les quatre conditions linéaires et leur signification pratique. Dans quelles conditions peut-on les réaliser ? Quelques solutions de l'équation de la chaleur, en coordonnées cartésiennes, cylindriques et sphériques avec les conditions linéaires. Cas des systèmes conductifs avec sources de chaleur. L'analogie électrique en stationnaire. Le problème de l'ailette rectangulaire longitudinale : Equation de l'ailette. Résolution. Calcul du rendement et de l'efficacité de l'ailette. Généralisation du concept d'ailette. Application { l'ailette radiale de profil uniforme.

Chapitre 4. Transfert de chaleur par convection (5 Semaines)
Mécanismes des transferts de chaleur par convection. Paramètres intervenant dans les transferts convectifs. Mise en évidence des différents types de transfert par convection : Convections forcée, naturelle et mixte. Citer des exemples courants. Discerner entre transfert convectif laminaire et turbulent dans les deux modes forcé et naturel. Méthodes de résolution d'un problème de convection (Analyse dimensionnelle et expériences, méthodes intégrales pour les équations approchées de couche limite, résolution des équations représentant la convection et analogie avec des phénomènes similaire comme les transferts de masse). Analyse dimensionnelle alliée aux expériences : Théorème Pi, faire apparaître les nombres sans dimensions les plus utilisés en convection (Reynolds, Prandtl, Grashoff, Rayleigh, Peclet et Nusselt) forcée et naturelle. Expliquer la signification de ces nombres.

Mode d'évaluation :
Contrôle continu : 40% ; Examen : 60%.

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