

People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

L'arbi Ben M'hidi Oum El Bouaghi University

Faculty of Exact Sciences and Natural and Life Sciences

Department of Mathematics

1st year university

Course:

Algebra 2

Produced by: Yamina Soula

University year: 2023-2024

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Introduction

This manuscript serves as a course support for Algebra 2 and Mathematical Tools. It is primarily intended for first-year mathematics students (both in the MI and ST tracks), as well as first-year students enrolled in the national biomedical engineering program. The main objective of this document is to provide a structured and accessible presentation of the fundamental notions required at this level of study, particularly in abstract algebra and its related tools.

This document is not intended to serve as a reference textbook under any circumstances. It should rather be considered as a learning aid or a concise memory support containing the essential concepts that every first-year student is expected to master. It is designed to accompany lectures and tutorials, helping students to consolidate their understanding through clear definitions, illustrative explanations, and carefully selected exercises. For this reason, each section is accompanied by exercises intended to reinforce theoretical concepts and develop mathematical reasoning skills. The manuscript is organized into five main chapters. The first chapter introduces the fundamental notions of mathematical logic and set theory, which constitute the basic language of modern mathematics. These notions are essential for formulating precise definitions, constructing rigorous proofs, and developing structured mathematical reasoning. A solid understanding of logic and sets is indispensable for progressing toward more advanced topics in algebra and analysis.

The second chapter is devoted to sets and functions, with emphasis on their basic properties and operations. Functions between sets are introduced as fundamental mathematical objects that allow the comparison and transformation of structures. This chapter provides the necessary background for understanding more abstract concepts encountered later in the course.

The third chapter focuses on binary relations defined on a set. This topic plays a central role in many areas of mathematics. Special attention is given to equivalence relations and partial orders, which allow the classification of elements and the description of hierarchical structures. Quotient sets and partition techniques are also introduced, providing a natural transition toward algebraic structures.

The fourth chapter presents the main algebraic structures, including groups, rings, and fields. These structures form one of the central pillars of modern algebra. The axiomatic approach is emphasized in order to highlight the role of operations and the fundamental properties they must satisfy. This unified framework allows different mathematical systems to be studied in a systematic and coherent manner.

The fifth chapter is dedicated to polynomial rings, which constitute an important class of algebraic structures with numerous applications in pure and applied mathematics. Polynomial arithmetic, divisibility, factorization, and related properties are studied in detail. These concepts play a fundamental role in later courses such as algebraic number theory, coding theory, and various areas of applied mathematics.

In addition to theoretical content, the manuscript includes a collection of exercises at the end of each section. These exercises are carefully designed to reinforce understanding and to develop the student's ability to apply abstract concepts in concrete situations. Some of them are provided with full solutions, while others are left as independent practice in order toencourage autonomy and critical thinking.

From a historical perspective, algebraic structures did not emerge as isolated concepts. Instead, they developed gradually from several branches of mathematics, including number theory, geometry, and analysis. This historical evolution explains the strong interconnections between algebra and other mathematical disciplines. Over time, mathematicians observed that many mathematical systems share similar structural properties, which led to the development of the abstract theory of algebraic structures.

A wide variety of algebraic structures exist, each defined by specific axioms. Among the most important are groups, rings, fields, vector spaces, and modules. Although these structures differ in their definitions, they share a common philosophy: they are based on sets equipped with one or more operations satisfying given properties. This unified viewpoint allows mathematicians to study them in a general and systematic way.

For these structures, the notion of homomorphism plays a central role. A homomorphism is a structure-preserving map between two algebraic systems, allowing the comparison of different structures within a common framework. In addition, important constructions naturally arise in this context, such as substructures, quotient structures, and direct products. These constructions enable the creation of new algebraic systems from existing ones while preserving essential properties.

One of the key features of algebraic structures is that many of their properties behave in a similar manner across different contexts. For instance, intersections of subgroups, subrings, or subspaces often preserve structure, and the image of a structure under a homomorphism typically retains algebraic properties. These similarities justify the use of a unified and abstract approach, which allows general results to be applied across multiple mathematical settings.

For this reason, algebraic structures are introduced within a general and abstract framework. This approach enables a unified treatment of constructions and properties, avoiding repetition across different mathematical contexts. Once the general theory is established, attention can then be directed toward the specific properties of each structure depending on its mathematical setting and applications.

The fifth chapter also includes a set of examination-style problems, designed to familiarize students with assessment methods commonly used in written examinations. Unlike previous chapters, which focus mainly on theoretical foundations, this section emphasizes practical evaluation and problem-solving strategies.

Each exercise is accompanied by a complete and detailed solution, with particular attention given to the logical structure of the answers. Step-by-step reasoning is clearly presented in order to reflect the expectations of formal examinations. In addition, a grading scheme is provided for each problem, indicating how marks are distributed among the different parts of the solution. This allows students to better understand evaluation criteria and to identify the essential steps required to obtain full credit.

The inclusion of marking guidelines is intended to help students become familiar with assessment standards and to develop effective strategies for writing structured and rigorous answers under exam conditions. By studying both solutions and grading distribution, students can gain deeper insight into the expectations of instructors and improve their performance in evaluations.

Overall, this manuscript aims not only to provide theoretical knowledge, but also to serve as a comprehensive training tool that supports both learning and assessment preparation. It is intended to strengthen students' mathematical foundations, develop their analytical thinking, and prepare them for more advanced studies in mathematics and related disciplines.

Chapter 1

Vector spaces

1.1 Introduction

This chapter introduces the fundamental concept of vector spaces, which is one of the central structures in linear algebra. Vector spaces provide a general framework for studying mathematical objects that can be added and multiplied by scalars, extending the classical notion of geometric vectors to more abstract settings. This abstraction allows many mathematical entities, such as matrices, functions, and sequences, to be treated within a unified structure. The study of vector spaces is essential for understanding linear relationships and forms the basis for many developments in modern mathematics and applied sciences. In particular, it provides the tools needed to analyze structure, dependence, and dimension in a rigorous way. The notion of linear independence and basis plays a key role in describing vector spaces efficiently and in introducing coordinates in abstract settings.
This chapter aims to establish the basic theoretical foundations required for working with vector spaces and their main properties, which will be used throughout the remainder of the course.

1.2 Vector Space and Subspace

Definition 1.2.1

Let K be a commutative field ($K = \mathbb{R}$ or $K = \mathbb{C}$), and let the non-empty set E be equipped with an internal operation denoted by $(+)$:

$$\begin{array}{rccl} (+): & E \times E & \to & E \\ & (x, y) & \mapsto & x + y, \end{array}$$

and also equipped with an external operation denoted by $(\,\cdot\,)$:

$$\begin{array}{rccl} (\,\cdot\,): & \mathbb{K} & \times E \to & E \\ & (\lambda, x) & \mapsto & \lambda \cdot x \end{array}$$

Definition 1.2.2

A vector space over the field $\mathbb{K}$ or an $\mathbb{K}$-vector space is a triplet $(E, +, \cdot)$ such that:

1. $(E, +)$ is a commutative group.
2. $\forall \lambda \in \mathbb{K}, \forall x, y \in E, \lambda(x + y) = \lambda \cdot x + \lambda \cdot y$.
3. $\forall \lambda, \mu \in \mathbb{K}, \forall x \in E, (\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$.
4. $\forall \lambda, \mu \in \mathbb{K}, \forall x \in E, (\lambda \cdot \mu) \cdot x = \lambda(\mu \cdot x)$.
5. $\forall x \in E, 1_{\mathbb{K}} \cdot x = x$.

The elements of K are called scalars, and those of the vector space are called vectors.

Proposition 1.2.3

If the set E is an $K-$vector space, then it satisfies the following properties:

1. $\forall x \in E,\ 0_{\mathbb{K}} \cdot x = 0_E$.
2. $\forall x \in E,\ -1_{\mathbb{K}} \cdot x = -x$.
3. $\forall \lambda \in \mathbb{K},\ \lambda \cdot 0_E = 0_E$.
4. $\forall \lambda \in \mathbb{K},\ \forall x, y \in E,\ \lambda \cdot (x - y) = \lambda \cdot x - \lambda \cdot y$.
5. $\forall \lambda \in \mathbb{K},\ \forall x \in E,\ x \cdot \lambda = 0_E \Leftrightarrow x = 0_E \vee \lambda = 0_{\mathbb{K}}$.

Example 1.2.4

1. $(\mathbb{R}, +, \cdot)$ is an $\mathbb{R}$-vector space, $(\mathbb{C}, +, \cdot)$ is a $\mathbb{C}$-vector space.
2. If we consider the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with the following operations:

$$\begin{array}{rccc} (+): & \mathbb{R}^2 \times \mathbb{R}^2 & \to \mathbb{R}^2 \\ & ((x,y),(x',y')) & \mapsto (x+x', y+y'). \end{array}$$

$$\begin{array}{rccc} (\cdot): & \mathbb{R} \times \mathbb{R}^2 & \to \mathbb{R}^2 \\ & (\lambda,(x,y)) & \mapsto (\lambda \cdot x, \lambda \cdot y). \end{array}$$

So, ($\mathbb{R}^2$, +, .) can be easily shown as an $\mathbb{R}-$vector space

Definition 1.2.5

Let F be a non-empty subset of E, and (E, +, .) be an $K-$vector space. We say that F is a subspace if (F,+, .) is also an $K-$vector space.

Note 1.2.6

- When $(F, +, \cdot)$ is an $\mathbb{K}$-subspace of $(E, +, \cdot)$, then $0_E \in F$.
- If $0_E \notin F$, then $(F, +, \cdot)$ cannot be an $\mathbb{K}$-subspace of $(E, +, \cdot)$.

Theorem 1.2.7

Let $(E, +, \cdot)$ be an $\mathbb{K}$-vector space and $F \subset E, F$ non-empty. The following are equivalent:

1. F is a subspace of E.
2. F is stable under addition and multiplication, i.e.,

$$\forall x, y \in F, \forall \lambda \in \mathbb{K}, x + y \in F, \lambda \cdot x \in F.$$

3. $\forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}, \lambda \cdot x + \mu \cdot y \in F$, i.e., F is a subspace $\Leftrightarrow \begin{cases} F \neq \phi, \\ \forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}, \lambda \cdot x + \mu \cdot y \in F \end{cases}$
4. $\forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}, \lambda \cdot x + \mu \cdot y \in F$, i.e., F is a subspace $\Leftrightarrow \begin{cases} 0_E \in F, \\ \forall x, y \in F, \forall \lambda, \mu \in \mathbb{K}, \lambda \cdot x + \mu \cdot y \in F \end{cases}$

Example 1.2.8

1. $\{0_E\}$ and E are subspaces of E.
2. $F = \{(x, y) \in \mathbb{R}^2 \mid x + y = 0\}$ is a subspace because,
   (a) $0_E = 0_{\mathbb{R}^2} = (0, 0) \in F$, then $F \neq \phi$
   (b) $\forall (x, y), (x', y') \in F, \lambda, \mu \in \mathbb{R}$, we want to prove that $\lambda(x, y) + \mu(x', y') \in F$, i.e., $(\lambda x + \lambda y) + (\mu x' + \mu y') = 0$, which holds true.
   Therefore, $\lambda(x, y) + \mu(x', y') \in F$, and F is a subspace of $\mathbb{R}^2$.
3. The set represented by: $F = \{(x + y + z, x - y, z) \mid x, y, z \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$ because,
   (a) $0_{\mathbb{R}^3} = (0, 0, 0) \in F$ since $(0, 0, 0) = (0 + 0 + 0, 0 - 0, 0) \Rightarrow F \neq \phi$.
   (b) $\forall X, Y \in F, \lambda, \mu \in \mathbb{R}$, we want to prove that $\lambda X + \mu Y \in F$. Let $X = (x, y, z)$ and $Y = (x', y', z')$. Then,

$$\begin{aligned} \lambda X + \mu Y &= (\lambda x + \lambda y + \lambda z, \lambda x - \lambda y, \lambda z) + (\mu x' + \mu y' + \mu z', \mu x' - \mu y', \mu z'), \\ &= ((\lambda x + \lambda y + \lambda z) + (\mu x' + \mu y' + \mu z'), (\lambda x - \lambda y) + (\mu x' - \mu y'), \lambda z + \mu z'), \\ &= ((\lambda x + \mu x') + (\lambda y + \mu y') + (\lambda z + \mu z'), (\lambda x - \mu y') + (\mu x' - \mu y'), \lambda z + \mu z'). \end{aligned}$$

   Therefore, $\lambda X + \mu Y \in F$. So, F is a subspace of $\mathbb{R}^3$.

Theorem 1.2.9

The intersection of a non-empty families of subspaces are defined as the subspace.

Proof

Let $(V, +, \cdot)$ be a vector space, and let $\{W_i\}_{i \in I}$ be a non-empty family of subspaces of V. Then the intersection

$$W = \underset{i \in I}{\cap} Wi$$

is also a subspace of V
To show that W is a subspace of V, we must verify three properties:

1. W is non-empty
   Since each W_i is a subspace, every subspace contains the zero vector 0.
   So:

$$0 \in W_i \quad \text{for all} \quad i \in I$$

   Hence:

$$0 \in \underset{i \in I}{\cap} Wi = W$$

   So

$$W \neq \emptyset$$

2. Stability under addition
   Let
   $u, v \in W$
   This means:

$$u \in W_i \quad \text{and} \quad v \in W_i \quad \text{for all} \quad i \in I$$

   Since each W_i is a subspace, it is closed under addition, so:

$$u + v \in Wi \quad \text{for all} \quad i \in I$$

Therefore:

$$u + v \in \bigcap_{i \in I} Wi = W$$

So

W is closed under addition.

3. Stability under scalar multiplication

Let

$u \in W$ and $\lambda \in \mathbb{R}$

Then:

$$u \in Wi \quad \text{for all} \quad i \in I$$

Since each

W_i is a subspace:

$$\lambda u \in Wi \quad \text{for all} \quad i \in I$$

Thus:

$$\lambda u \in \bigcap_{i \in I} Wi = W$$

So,

W is closed under scalar multiplication.

Example 1.2.10

Consider the vector space $\mathbb{R}^2$, and define two subspaces:

W_1: the $x-$axis

$$W_1 = \{(x, 0) \quad : \quad x \in \mathbb{R}\}$$

W_2: the $y-$axis

$$W_2 = \{(0, y) \quad : \quad y \in \mathbb{R}\}$$

The intersection consists of all vectors that belong to both subspaces:

- A vector in W_1 must be of the form $(x, 0)$
- A vector in W_2 must be of the form $(0, y)$

For a vector to belong to both, it must satisfy:

$$(x, 0) = (0, y)$$

This implies:

$$x = 0 \quad \text{and} \quad y = 0$$

Hence:

$$W_1 \cap W_2 = \{(0, 0)\}$$

Theorem 1.2.11

The union of two subspaces is not always a subspace.

Proof

Proof (by counterexample)

We will show that the union of two subspaces may fail to be closed under addition.
Consider $\mathbb{R}^2$
Let:
$W_1 = \{(x, 0) \ : \ x \in \mathbb{R}\}$
$W_2 = \{(0, y) \ : \ y \in \mathbb{R}\}$
Both W_1 and W_2 are subspaces of $\mathbb{R}^2$

$$W_1 \cup W_2$$

contains all vectors that lie either on the $x-$axis or on the $y-$axis.
So for example:

$$u = (1, 0) \in W_1 \subset W_1 \cup W_2$$

$$v = (0, 1) \in W_2 \subset W_1 \cup W_2$$

Now add the two vectors:

$$u + v = (1, 0) + (0, 1) = (1, 1)$$

But:
• $(1, 1)$ is not in W_1 (second coordinate is not zero)
• $(1, 1)$ is not in W_2 (first coordinate is not zero)
So,

$$(1, 1) \notin W_1 \cup W_2$$

1.2.1 Sum of Two Subspaces

Definition 1.2.12

Let a vector space E consist of two subspaces, E_1 and E_2. The sum of the two vector spaces denoted by $E_1 + E_2$, is defined as follows:

$$E_1 + E_2 = \{U \in E \mid \exists U_1 \in E_1, \ \exists U_2 \in E_2 \text{ such that } U = U_1 + U_2\}.$$

Proposition 1.2.13

The sum of two subspaces E_1 and E_2 of the same vector space E is a subspace of E containing $E_1 \cup E_2$.

1.2.2 Direct Sum of Two Subspaces

Definition 1.2.14

The sum $E_1 + E_2$ is said to be direct if, for every $U = U_1 + U_2$, there exists a unique vector $U_1 \in E_1$ and a unique vector $U_2 \in E_2$ such that $U = U_1 + U_2$. This is denoted as $E_1 \oplus E_2$.

Theorem 1.2.15

The sum $E_1 + E_2$ is direct if and only if $E_1 \cap E_2 = \{0_E\}$.

1.2.3 Generating Sets, Linearly Independent Sets, Bases, and Dimension

In the following, we denote the vector space $(E, +, .)$ as E.

Definition 1.2.16

Let E be a vector space and $e_1, e_2, ..., e_n$ be elements of E.

1. The set $\{e_1, e_2, ..., e_n\}$ is called linearly independent if for all $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$:

$$\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n = 0 \Rightarrow \lambda_1 = \lambda_2 = ... = \lambda_n = 0,$$

   otherwise, they are called dependent.

2. The set $\{e_1, e_2, ..., e_n\}$ is called a generating set of E or E is generated by $\{e_1, e_2, ..., e_n\}$ if for every $x \in E$, there exist $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$ such that $x = \lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n$.

3. If $\{e_1, e_2, ..., e_n\}$ is a linearly independent and generating set of E, then it is called a basis of E.

Note 1.2.17

In a vector space E, every nonzero vector is linearly independent.

Theorem 1.2.18

If $\{e_1, e_2, ..., e_n\}$ and $\{e_1^{'}, e_2^{'}, ..., e_m^{'}\}$ are two bases of the vector space E, then $n = m$. In other words, if a vector space has a basis, then all bases of E have the same number of elements (or the same cardinality), and this number depends only on the vector space E.

Definition 1.2.19

If E is an $\mathbb{R}$-vector space with basis $B = \{e_1, e_2, ..., e_n\}$, then the dimension of E is defined as:

$$\dim(E) = \text{Card}(B).$$

Note 1.2.20

Finding a set of vectors in E that form a linearly independent basis for a vector space is known as basis. and generating set for E. The number of elements in this set represents $\dim(E)$.

Example 1.2.21

Find a basis for $\mathbb{R}^3$. We can use the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$.
A basis of $\mathbb{R}^3$
is a set of 3 linearly independent vectors that span the whole space.
The standard basis is:

$$e_1 = (1,0,0), \qquad e_2 = (0,1,0), \qquad e_3 = (0,0,1)$$

These vectors are:
Linearly independent (none can be written as a combination of the others)
They generate all vectors in $\mathbb{R}^3$.

Example 1.2.22

Show that $f_1 = (1, -1), f_2 = (1, 1)$ form a basis for $\mathbb{R}^2$. Show that $\{f_1, f_2\}$ is linearly independent and generating.

Step 1: Linear independence

Assume:

$$a(1, -1) + b(1, 1) = (0, 0)$$

This gives:

$$(a + b,\ -a + b) = (0, 0)$$

So:

$$\begin{cases} a + b = 0 \\ -b + a = 0 \end{cases}$$

Solve:

- From $a + b = 0 \Rightarrow b = -a$
- Substitute into second: $-a + (-a) = 0 \Rightarrow -2a = 0 \Rightarrow a = 0$
- Then $b = 0$.

So vectors are linearly independent.

Step 2: They span $\mathbb{R}^2$

Take any $(x, y) \in \mathbb{R}^2$. We try:

$$xf_1 + yf_2 \text{ style combination}$$

Instead solve:

$$a(1, -1) + b(1, 1) = (x, y)$$

This gives:

$$\begin{cases} a + b = x \\ b - a = y \end{cases}$$

Add equations:

$$2b = x + y \Rightarrow b = \frac{x + y}{2}$$

Subtract:

$$2a = x - y \Rightarrow a = \frac{x - y}{2}$$

So every vector is generated.

Example 1.2.23

Consider $F = \{(x + y, x - z, -y - z) \mid x, y, z \in \mathbb{R}\}$. Find a basis for F. **Step 1:** Write as a linear combination

$$(x + y,\ x - z,\ -y - z)$$

Split by variables:

- Terms with x : $(x, x, 0) = x(1, 1, 0)$
- Terms with y : $(y, 0, -y) = y(1, 0, -1)$
- Terms with z : $(0, -z, -z) = z(0, -1, -1)$

So:

$$F = \text{Span}\{(1, 1, 0), (1, 0, -1), (0, -1, -1)\}$$

Step 2: Check dependence

We test quickly:

Notice:

$$(1,1,0)-(1,0,-1)=(0,1,1)$$

But:

$$(0,-1,-1)=-(0,1,1)$$

So the third vector is dependent.
Thus we only need two independent vectors.
Take:

$$(1,1,0),(1,0,-1)$$

They are linearly independent.
Final answer: basis of F is:

$$\{(1,1,0),\quad (1,0,-1)\}$$

1.2.4 Vector Space of Finite Dimension (Properties)

Theorem 1.2.24

Let E be a vector space of dimension n:

1. The set $\{e_1,e_2,...,e_n\}$ is a basis of E if and only if $\{e_1,e_2,...,e_n\}$ is generating and linearly independent.
2. If $\{e_1,e_2,...,e_p\}$ is a set of p vectors in E with $p>n$, then $\{e_1,e_2,...,e_p\}$ cannot be linearly independent. Moreover, if $\{e_1,e_2,...,e_p\}$ is generating, then there exist n vectors among $\{e_1,e_2,...,e_p\}$ that form a basis for E.
3. If $\{e_1,e_2,...,e_p\}$ is a set of p vectors in E with $p<n$, then $\{e_1,e_2,...,e_p\}$ cannot be generating. Moreover, if $\{e_1,e_2,...,e_p\}$ is linearly independent, then there exist $(n-p)$ vectors among $\{e_{p+1},e_{p+2},...,e_n\}$ in E such that $\{e_1,e_2,...,e_p,e_{p+1},e_{p+2},...,e_n\}$ is a basis for E.

Example 1.2.25

1. In the previous example, $f_1=(1,-1)$, $f_2=(2,1)$ to show that $\{f_1,f_2\}$ forms a basis of $\mathbb{R}^2$, it suffices to show that $\{f_1,f_2\}$ is either linearly independent or generating (this property holds in the case of finite-dimensional vector spaces).
2. To show that $\{f_1=(1,1,1),\ f_2=(1,1,0),\ f_3=(0,1,-1)\}$ is a basis of $\mathbb{R}^3$, it suffices to show that it is either linearly independent or generating since $dim\ \mathbb{R}^3=3$. $\{f_1,f_2,f_3\}$ is linearly independent because: For all $\lambda_1,\ \lambda_2,\ \lambda_3\in\mathbb{R}$:

   $$\lambda_1(1,1,1)+\lambda_2(1,1,0)+\lambda_3(0,1,-1)=(0,0,0)\Rightarrow\lambda_1=\lambda_2=\lambda_3=0,$$

   which implies that $\{(1,1,1),(1,1,0),(0,1,-1)\}$ is a basis of $\mathbb{R}^3$.
3. Let's find a basis for $F=\{(x+y,x-z,-y-z)\mid x,y,z\in\mathbb{R}\}$. Since $F\subset\mathbb{R}^3$, $dimF\leq 3$, so the basis of F does not have more than three vectors. $(x+y,x-z,-y-z)=x(1,1,0)+y(1,0,-1)+z(0,-1,-1)$. Thus, $v_1=(1,1,0)$, $v_2=(1,0,-1)$, $v_3=(0,-1,-1)$ form a generating family for F. If this family is linearly independent, then it forms a basis for F.
   For all $\lambda_1,\ \lambda_2\ ,\lambda_3\in\mathbb{R}$,

   $$\lambda_1(1,1,0)+\lambda_2(1,0,-1)+\lambda_3(0,-1,-1)=(0,0,0)\Rightarrow\lambda_2=-\lambda_1,\lambda_1=\lambda_3.$$

   So, $\{(1,1,0),(1,0,-1),(0,-1,-1)\}$ is not linearly independent. However, according to the previous theorem, we can extract from this family a basis for F. To do this, we need to find two vectors from the family that are linearly independent. If found, they form a basis for F; otherwise, we take a non-null vector, and it becomes the basis for F. For example, let's take $\{v_1,v_2\}$:

   $$\lambda_1(1,1,0)+\lambda_2(1,0,-1)=(0,0,0)\Rightarrow\lambda_1=\lambda_2=0.$$

- Thus, $\{v_1, v_2\}$ is a basis for F and $dimF = 2$.

1.2.5 Supplementary Vector Subspace

Definition 1.2.26

Assume that vector space E has two subspaces, E_1 and E_2.
If $E_1 \oplus E_2 = E$, then E_1 and E_2 are considered supplemental.

Example 1.2.27

Let $E_1 = \{(x, 0) \in \mathbb{R}^2\}$, and $E_2 = \{(0, y) \in \mathbb{R}^2\}$.
Note that: $E_1 \oplus E_2 = \mathbb{R}^2$, then E_1 and E_2 are supplementary.

1.3 Exercises

Exercise 1.3.1

1. Consider in $\mathbb{R}^3$ the subset F defined by:
$$F = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - z = 0\}.$$
Show that F is a subspace of $\mathbb{R}^3$.
2. Give a basis for F, and what is its dimension?
3. Is F equal to $\mathbb{R}^3$?

Exercise 1.3.2

Consider in $\mathbb{R}^3$ the subset F defined by:
$$F = \{(x - y, 2x + y + 4z, 3y + 2z) \mid x, y, z \in \mathbb{R}\}.$$
1. Show that F is a subspace of $\mathbb{R}^3$.
2. Give a basis for F, and what is its dimension?
3. Is F equal to $\mathbb{R}^3$?

Exercise 1.3.3

Consider in $\mathbb{R}^4$ the subset F defined by:
$$F = \{(x, y, z, t) \in \mathbb{R}^4 \mid (x + z = 0) \wedge (y + t = 0)\}.$$
1. Show that F is a subspace of $\mathbb{R}^4$.
2. Give a basis for F, and deduce its dimension.

Exercise 1.3.4

1. Show that the family $\{(1, 2), (-1, 1)\}$ generates $\mathbb{R}^2$.
2. Which families are free among the following: $F_1 = \{(1, 1, 0), (1, 0, 0), (0, 1, 1)\}$, $F_2 = \{(0, 1, 1, 0), (1, 1, 1, 0), (2, 1, 1, 0)\}$?
3. Show that the family $\{(1, 2), (-1, 1)\}$ is a basis for $\mathbb{R}^2$, and that the family $F_1 = \{(1, 1, 0), (1, 0, 0), (0, 1, 1)\}$ is a basis for $\mathbb{R}^3$.

1.4 Solution of exercises

Solution 1.3.1

1. F is a subspace vector space (s.e.v.) if and only if:

$$\begin{cases} F \neq \phi \\ \forall X, Y \in F, \forall \lambda, \mu \in \mathbb{R},\ \lambda X + \mu Y \in F. \end{cases}$$

   (a) $0_{\mathbb{R}^3} = (0,0,0) \in F \Leftrightarrow F \neq \phi$ because $2 \cdot 0 + 0 - 0 = 0$.

   (b) For all $X = (x,y,z)$, $Y = (x',y',z') \in F$, and $\lambda, \mu \in \mathbb{R}$, we want to show that:

$$\lambda(x,y,z) + \mu(x',y',z') \overset{?}{\in} F,$$

   i.e.,

$$(\lambda x + \mu x', \lambda y + \mu y', \lambda z + \mu z') \overset{?}{\in} F.$$

   $2(\lambda x + \mu x') + (\lambda y + \mu y') - (\lambda z + \mu z') = \lambda(2x + y - z) + \mu(2x' + y' - z') = \lambda \cdot 0 + \mu \cdot 0 = 0$,
   since $(x,y,z) \in F \Rightarrow 2x + y - z = 0$ and $(x',y',z') \in F \Rightarrow 2x' + y' - z' = 0$.
   Thus, $\lambda(x,y,z) + \mu(x',y',z') \in F$, and F is a subspace vector space of $\mathbb{R}^3$.

2. Basis of F: Let $X \in F$, i.e., $2x + y - z = 0 \Rightarrow z = 2x + y$. $X = (x,y,z) = (x,y,2x+y) = x(1,0,2) + y(0,1,1)$, so

$$F = \{(x,y,z) \in \mathbb{R}^3 \mid 2x + y - z = 0\} = \{x(1,0,2) + y(0,1,1) \mid x, y \in \mathbb{R}\}.$$

   Hence, F is generated by $\{v_1 = (1,0,2), v_2 = (0,1,1)\}$. Let's show that this family is linearly independent if and only if:

$$\forall \lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 v_1 + \lambda_2 v_2 = (0,0,0) \Rightarrow \lambda_1 = \lambda_2 = 0.$$

   $\lambda_1(1,0,2) + \lambda_2(0,1,1) = (0,0,0) \Rightarrow (\lambda_1, \lambda_2, 2\lambda_1 + \lambda_2) = (0,0,0)$.
   This implies $\lambda_2 = -\lambda_1$ and $\lambda_1 = \lambda_2 = 0$.
   Therefore, $\{v_1, v_2\}$ is a basis (linearly independent and generating) of $\mathbb{R}^3$.

3. $F \neq \mathbb{R}^3$ since $\dim F = 2 \neq 3 = \dim \mathbb{R}^3$.

Solution 1.3.2

1. $(0,0,0) \in F$ because $(0,0,0) = (0-0, 2\times 0 + 0 + 4 \times 0, 3 \times 0 + 2 \times 0)) \Rightarrow F \neq \phi$.

   $\forall X, Y \in F$, λ, $\mu \in \mathbb{R}$, let's show that $\lambda X + \mu Y \overset{?}{\in} F$
   $X \in F,\ \exists (x,y,z) \in \mathbb{R}^3 /\quad X = (x - y, 2x + y + 4z, 3y + 2z)$
   $Y \in F,\ \exists (x',y',z') \in \mathbb{R}^3 /\quad Y = (x' - y', 2x' + y' + 4z', 3y' + 2z')$
   $\lambda\ X + \mu Y = (\lambda x - \lambda y + \mu x' - \mu y', 2\lambda x + \lambda y + 4\lambda z + 2\mu x' + \mu y' + 4\mu z', 3\lambda y + 2\lambda z + 3\mu y' + 2\mu z') +$
   $(\lambda x - \lambda y + \mu x' - \mu y', 2\lambda x + \lambda y + 4\lambda z + 2\mu x' + \mu y' + 4\mu z', 3\lambda y + 2\lambda z + 3\mu y' + 2\mu z')$
   $= ((\lambda x + \mu x') - (\lambda y + \mu y'), 2(\lambda x + \mu x') + (\lambda y + \mu y') + 4(\lambda z + \mu z'), 3(\lambda y + 2\mu z) + 2\lambda z + 2\mu z')$
   $= ((\lambda x + \mu x') - (\lambda y + \mu y'), 2(\lambda x + \mu x') + (\lambda y + \mu y') + 4(\lambda z + \mu z'), 3(\lambda y + 2\mu z) + 2(\lambda + \mu) z')$
   $\exists x'' = (\lambda x + \mu x')$, $\exists y'' = (\lambda y + \mu y')$, $\exists z'' = (\lambda z + \mu z')$, thus
   $\lambda X + \mu Y = (x'' - y'',\ 2x'' + y'' + 4z'',\ 3y'' + 2z'') \in F$.

2. Basis of F:
   Let $X \in F$, $\exists (x,y,z) \in \mathbb{R}^3 /\quad X = (x - y,\ 2x + y + 4z,\ 3y + 2z)$,
   $X = (x - y, 2x + y + 4z, 3y + 2z) = x(1,2,0) + y(-1,1,3) + y(0,4,2)$, thus
   $F = \{x(1,2,0) + y(-1,1,3) + y(0,4,2) / x, y \in \mathbb{R}\}$.
   Therefore, F is generated by $\{v_1 = (1,2,0), v_2 = (-1,1,3), v_3 = (0,4,2)\}$,

let's show that this family is linearly independent if and only if

$\forall\ \lambda_1,\lambda_2,\ \lambda_3 \in \mathbb{R},\ \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = (0,0,0) \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0.$

$\lambda_1(1,2,0) + \lambda_2(-1,1,3) + \lambda_3(0,4,2) = (0,0,0) \Rightarrow (\ \lambda_1,\ \lambda_2, -\lambda_1, -\lambda_2) = (0,0,0)$

Thus the dimension of F is 3, as $\{v_1, v_2, v_3\}$ is a basis (linearly independent and generating) of $\mathbb{R}^3$.

3. $F = \mathbb{R}^3$ because $\dim F = 3 = \dim \mathbb{R}^3$.

Solution 1.3.3

1. • $(0,0,0,0) \in F \Rightarrow F \neq \phi$ because $(0+0=0) \wedge (0+0=0)$.

   • $\forall X = (x,y,z,t),\ Y = (x',y',z',t') \in F,\ \lambda,\ \mu \in \mathbb{R}$, let's show that:

   $\lambda(x,y,z,t) + \mu(x',y',z',t') \overset{?}{\in} F$

   We have:

   $$\begin{cases} X \in F \Rightarrow (x+z=0) \wedge (y+t=0) \\ Y \in F \Rightarrow (x'+z'=0) \wedge (y'+t'=0) \end{cases}$$

   $$\Rightarrow \begin{cases} \lambda(x+z) = 0 \wedge \mu(x'+z') = 0 \\ \wedge \\ \lambda(y+t) = 0 \wedge \mu(y'+t') = 0 \end{cases}$$

   $$\Rightarrow \begin{cases} \lambda x + \mu x' + \lambda z + \mu z' = 0 \\ \wedge \\ \lambda y + \mu t + \lambda t + \mu t' = 0 \end{cases}$$

   Therefore:

   $(\lambda x + \mu x' + \lambda z + \mu z') = 0 \wedge (\lambda y + \mu t + \lambda t + \mu t') = 0$

   which means

   $\lambda(x,y,z,t) + \mu(x',y',z',t') \in F$ hence the result.

2. Basis of F:
   For $X \in F,\ x = -z \wedge y = -t$,

   $X = (x,y,z,t) = (x,y,-x,-y) = x(1,0,-1,0) + y(0,1,0,-1)$

   $F = \{x(1,0,-1,0) + y(0,1,0,-1) / x,y \in \mathbb{R}\}$.

   Therefore, F is generated by $\{v_1 = (1,0,-1,0),\ \ v_2 = (0,1,0,-1)\}$, let's show that this family is linearly independent if and only if

   $\forall\ \lambda_1,\ \lambda_2 \in \mathbb{R},\ \lambda_1 v_1 + \lambda_2 v_2 = (0,0,0,0) \Rightarrow \lambda_1 = \lambda_2 = 0.$

   $\lambda_1(1,0,-1,0) + \lambda_2(0,1,0,-1) = (0,0,0,0) \Rightarrow (\ \lambda_1,\ \lambda_2, -\lambda_1, -\lambda_2) = (0,0,0,0)$

   Therefore, the dimension of F is 2, as $\{v_1, v_2\}$ is a basis (linearly independent and generating) of $\mathbb{R}^4$.

Solution 1.3.4

1. The family $\{(1,2),(-1,1)\}$ generates $\mathbb{R}^2$ if and only if
   For every $X = (x,y) \in \mathbb{R}^2$, there exist $\lambda, \mu \in \mathbb{R}$ such that $X = \lambda(1,2) + \mu(-1,1)$.

   Let $(x,y) \in \mathbb{R}^2$, let's find $\lambda, \mu \in \mathbb{R}$ such that:

   $(x,y) = \lambda(1,2) + \mu(-1,1) = (\lambda - \mu, 2\lambda + \mu)$

   So we have:

   $$\begin{cases} x = \lambda - \mu \quad (1) \\ y = 2\lambda + \mu \quad (2) \end{cases}$$

   $\Rightarrow \lambda = \frac{x+y}{3}$ and $\mu = \frac{-2x+y}{3}$.

   Hence, this family is generating.

2. Which among the following families are linearly independent:

   $F_1 = \{(1,1,0),(1,0,0),(0,1,1)\}$,

$F_2 = \{(0,1,1,0),(1,1,1,0),(2,1,1,0)\}$.

(a) $F_1 = \{(1,1,0),(1,0,0),(0,1,1)\}$ is independent if and only if
For all λ_1, λ_2, $\lambda_3 \in \mathbb{R}$, $\lambda_1(1,1,0)+\lambda_2(1,0,0)+\lambda_3(0,1,1)=(0,0,0)$

$$\begin{cases} \lambda_1+\lambda_2=0 \\ \lambda_1+\lambda_3=0 \\ \lambda_3=0 \end{cases} \Rightarrow \lambda_1=\lambda_2=\lambda_3=0.$$

Therefore, F_1 is linearly independent.

(b) $F_2 = \{(0,1,1,0),(1,1,1,0),(2,1,1,0)\}$ is not independent because
There exist $\lambda_1=1$, $\lambda_2=-2$, $\lambda_3=1$ in $\mathbb{R}$ such that $\lambda_1(0,1,1,0)+\lambda_2(1,1,1,0)+\lambda_3(2,1,1,0)=(0,0,0,0)$.

3. The family $\{(1,2),(-1,1)\}$ is a basis for $\mathbb{R}^2$, because when the number of vectors $= 2 = \dim \mathbb{R}^2$, it is sufficient to show that it is either generating or independent for it to be a basis. According to question (1), it is generating.
The family $F_1 = \{(1,1,0),(1,0,0),(0,1,1)\}$ is a basis for $\mathbb{R}^3$, because the cardinality of F_1 is equal to $3 = \dim \mathbb{R}^3$, and F_1 is linearly independent, making it a basis for $\mathbb{R}^3$.

Chapter 2

Linear Applications

2.1 Introduction

This chapter introduces the concept of linear applications, which represent one of the most important tools in linear algebra. A linear application is a function between vector spaces that preserves the operations of addition and scalar multiplication, and it allows the study of transformations in a structured and algebraic way. This notion provides a natural link between abstract vector spaces and concrete computations using coordinates and matrices.
The study of linear applications is fundamental because it enables the analysis of how vector spaces are transformed while preserving their linear structure. Key notions such as the image and kernel of a linear map play a central role in understanding its behavior and in characterizing its properties. These concepts also lead naturally to the notion of rank, which measures the effective dimension of the image.
This chapter also introduces important theoretical results in finite-dimensional spaces, particularly the rank theorem, which establishes a fundamental relationship between the dimension of the domain, the kernel, and the image. Furthermore, the composition of linear applications and the study of bijective linear maps, including automorphisms, are discussed to provide a deeper understanding of the structure of linear transformations. Overall, this chapter aims to equip the student with the essential theoretical and computational tools needed to work with linear applications and their properties.

2.2 Concept of Linear Application

Definition 2.2.1

1. Assume that $(E,+,.)$ and $(F,+,.)$ are two K-vector spaces. Define f as a function that maps from E to F. f is considered a linear application if and only if:

$$\forall x,y \in E, \forall \lambda \in \mathbb{R}, \quad f(x+y) = f(x) + f(y) \quad \text{and} \quad f(\lambda x) = \lambda f(x)$$

   Or, in the equivalent case:

$$\forall x,y \in E, \forall \lambda,\mu \in \mathbb{R}, \quad f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$$

2. Furthermore, we refer to f as an isomorphism from E to F if f is bijective.
3. A linear application from $(E,+,.)$ to $(E,+,.)$ is called an endomorphism.
4. An isomorphism from $(E,+,.)$ to $(E,+,.)$ is also called an automorphism of E.

Example 2.2.2

The function

$$\begin{aligned} f_1 : \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto x - y, \end{aligned}$$

is a linear application, as:

$$\begin{aligned} f_1(\lambda(x,y)+\mu(x',y')) &= f_1((\lambda x,\lambda y)+(\mu x',\mu y')) \\ &= f_1((\lambda x+\mu x'),(\lambda y+\mu y')) \\ &= (\lambda x+\mu x')-(\lambda y+\mu y') \\ &= \lambda x-\mu y+\mu x'-\mu y', \\ &= \lambda(x-y)+\mu(x'-y'), \\ &= \lambda f_1(x,y)+\mu f_1(x',y'). \end{aligned}$$

Example 2.2.3

The function

$$\begin{aligned} f_2 : \mathbb{R}^3 &\to \mathbb{R}^3 \\ (x,y,z) &\mapsto (-x+y, x-5z, y), \end{aligned}$$

is a linear application, as:

$$\begin{aligned} f_2(\lambda(x,y,z)+\mu(x',y',z')) &= f_2((\lambda x,\lambda y,\lambda z)+(\mu x',\mu y',\mu z')) \\ &= f_2((\lambda x+\mu x'),(\lambda y+\mu y'),(\lambda z+\mu z')) \\ &= (-(\lambda x+\mu x')+(\lambda y+\mu y'),(\lambda x+\mu x')-5(\lambda z+\mu z'),\lambda y+\mu y') \\ &= (-\lambda x-\mu x'+\lambda y+\mu y',\lambda x+\mu x'-5\lambda z-5\mu z',\lambda y+\mu y') \\ &= (-\lambda x+\lambda y-\mu x'+\mu y',\lambda x-5\lambda z+\mu x'-5\mu z',\lambda y+\mu y') \\ &= (\lambda(-x+y)+\mu(-x'+y'),\lambda(x-5z)+\mu(x'-5z'),\lambda y+\mu y') \\ &= \lambda(-x+y,x-5z,y)+\mu(-x'+y',x'-5z',y'), \\ &= \lambda f_2(x,y,z)+\mu f_2(x',y',z'). \end{aligned}$$

Example 2.2.4

The function

$$\begin{aligned} f_3 : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto -5x, \end{aligned}$$

is an isomorphism. Indeed, f_3 is linear because for all x, y in $\mathbb{R}$, and λ, μ in $\mathbb{R}$:

$$\begin{aligned} f_3(\lambda x+\mu y) &= -5(\lambda x+\mu y), \\ &= \lambda f_3(x)+\mu f_3(y). \end{aligned}$$

Note 2.2.5

It can be easily shown that the sum of two linear applications is linear. Also, the product of a linear application by a scalar and the composition of two linear applications are linear.

Proposition 2.2.6

Let f be a linear application from E to F.

1. $f(O_E) = O_F$.
2. $\forall x \in E$, $f(-x) = -f(x)$.

Proof

Because f is a linear application we have:

1. $f(O_E) = f(O_E + O_E) = f(O_E) + f(O_E) \Rightarrow f(O_E) - f(O_E) = f(O_E)$. So, $O_F = f(O_E)$.

2. Since f is a linear application, then $\forall x \in E$, $f(-x) + f(x) = f(-x + x) = f(O_E) = O_F$ then, $f(-x) = -f(x)$.

2.2.1 Image and Kernel of a Linear Application

Definition 2.2.7

1. The definition of the image of f, or $Im\ f$, is as follows:

$$Im\ f = \{y \in F, \quad \exists x \in E : f(x) = y\} = \{f(x), \quad x \in E\}.$$

2. The following is the definition of the kernel of f, or $ker\ f$:

$$ker\ f = \{x \in E, \quad f(x) = O_F\}.$$

Sometimes $ker\ f$ is denoted by: $f^{-1}(\{0\})$.

2.2.2 Rank of a Linear Application

Proposition 2.2.8

If f is a linear application from E to F, and if $dim\ Im\ f = n < +\infty$, then n is called the rank of f and is denoted $rg(f)$. $Im\ f$ and $ker\ f$ are vector subspaces of E.

Definition 2.2.9

The dimension of the image of f is the definition of the rank of f:

$$rg(f) = dim\ Im\ f.$$

Example 2.2.10

1. Determine the kernel of the function f_1:

$$f_1(x, y) = x + 2y$$

The kernel is given by:

$$\begin{aligned} ker\ f_1 &= \{(x, y) \in \mathbb{R}^2 / f_1(x, y) = 0\} \\ &= \{(x, y) \in \mathbb{R}^2 / x + 2y = 0\} \\ &= \{(x, y) \in \mathbb{R}^2 / x = -2y\} \end{aligned}$$

So, $ker\ f_1$ is a line in $\mathbb{R}^2$.

2. Determine the kernel of the function f_2:

$$f_2(x, y, z) = (-x + y, x - 5z, y)$$

The kernel is given by:

$$\begin{aligned} ker\ f_2 &= \{(x, y, z) \in \mathbb{R}^3 / f_2(x, y, z) = (0, 0, 0)\} \\ &= \{(x, y, z) \in \mathbb{R}^3 / -x + y = 0, x - 5z = 0, y = 0\} \end{aligned}$$

Solving this system of equations gives the solution space, which is a subspace of $\mathbb{R}^3$.

Proposition 2.2.11

Let f be a linear application from E to F. The following are equivalent:

1. f is surjective $\Leftrightarrow Im\ f = F$.
2. f is injective $\Leftrightarrow ker\ f = \{0_E\}$.

Example 2.2.12

In the previous example $Im\ f_2 = \mathbb{R}^3$ then f_2 is surjective. We are going to demonstrate that f_2 is injective

$ker\ f_2 = \{(x,y,z) \in \mathbb{R}^3,\ \ f_2(x,y,z) = (0,0,0)\}$,

$ker\ f_2 = \{(x,y,z) \in \mathbb{R}^3, (-x+y, x-z, y) = (0,0,0)\}$

$$\begin{cases} -x+y=0 \\ x-z=0 \\ y=0 \end{cases} \Rightarrow \begin{cases} -x+0=0 \\ x-z=0 \\ y=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ z=0 \\ y=0 \end{cases}$$

Therefore, $ker\ f_2 = \{(0,0,0)\}$, hence f_2 is injective. So f_2 is bijective.

2.3 Finite dimension and rank theorem

Proposition 2.3.1

Let f, g be two linear mappings from E to F, such as E and F be two K vector spaces. If E is of finite dimension n and $\{e_1, e_2, ..., e_n\}$ is a basis of E. Then
$\forall k \in \{1,2,..,n\},\ f(e_k) = g(e_k) \Leftrightarrow \forall x \in E,\ f(x) = g(x)$.

Proof

The implication ($\Leftarrow$) is obvious. For ($\Rightarrow$)
Since E is generated by $\{e_1, e_2, ..., e_n\}$, for any $x \in E$, there exist λ_1,λ_2, ..., $\lambda_n \in \mathbb{K}$ such that $x = \lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n$,
As f and g are linear, we have
$f(x) = f(\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n) = \lambda_1 f(e_1) + \lambda_2 f(e_2) + ... + \lambda_n f(e_n)$,
$g(x) = g(\lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_n e_n) = \lambda_1 g(e_1) + \lambda_2 g(e_2) + ... + \lambda_n g(e_n)$,
so if we assume that $\forall k \in \{1,2,..,n\},\ f(e_k) = g(e_k)$, then we deduce that
$\forall x \in E,\ f(x) = g(x)$.

Note 2.3.2

For two linear mappings f and g from E to F to be equal, it is sufficient that they coincide on the basis of the $\mathbb{K}$ vector space E.

Example 2.3.3

Let g be a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that
$g(1,0) = (2,1),\ \ g(0,1) = (-1,-1)$.
Then let's determine the value of g at all points of $\mathbb{R}^2$, in fact, we have:

$\forall (x,y) \in \mathbb{R}^2, (x,y) = x(1,0) + y(0,1)$.

$\forall (x,y) \in \mathbb{R}^2$,
$g(x,y) = g(x(1,0) + y(0,1)) = xg(1,0) + yg(0,1)$
$= x(2,1) + y(-1,-1)$
$= (2x,x) + (-y,-y)$

$= (2x - y, x - y)$.
So, $g(x, y) = (2x - y, x - y)$.

Theorem 2.3.4

(**Rank Theorem**)
Let E be a vector space of finite dimension and f a linear mapping from E to F. Next, there is the equality.

$$\boxed{dim\ (Im\ f\ ) + dim\ (\ Ker\ f\ ) = dim(E\ )}$$

Another way to express the rank theorem is as

$$\boxed{rg(f\ ) + dim\ (Ker\ f\ ) = dim(E\ )}$$

Proof

Step 1: Basis of the kernel
Let:

$$\ker(f) \subset E$$

Since
E is finite-dimensional, $\ker(f)$ is also finite-dimensional.
Let:

$$\{u_1, u_2, \ldots, u_k\}$$

be a basis of $\ker(f)$
So:

$$\dim(\ker f) = k$$

Step 2: Extend to a basis of E .
We extend this basis of $\ker(f)$ to a basis of the whole space E:

$$\{u_1, \ldots, u_k, v_1, \ldots, v_r\}$$

is a basis of E
So:

$$\dim(E) = k + r$$

Step 3: Apply the linear map f
Apply f to the basis vectors:
• For $u_i \in \ker(f)$, we have:

$$f(u_i) = 0$$

• For $v_1, \ldots, v_r$, consider:

$$f(v_1), \ldots, f(v_r)$$

Step 4: Show they generate $Im(f)$
Let $y \in Im(f)$. Then:

$$y = f(x), \quad x \in E$$

Write:

$$x = \sum_{i=1}^{k} a_i u_i + \sum_{j=1}^{r} b_j v_j$$

Apply f:

$$f(x) = \sum_{i=1}^{k} a_i \ f(u_i) + \sum_{j=1}^{r} b_j f(v_j)$$

Since $f(u_i) = 0$

$$f(x) = \sum_{j=1}^{r} b_j f(v_j)$$

So:

$$Im(f) = \text{Span}\{f(v_1), \ldots, f(v_r)\}$$

Step 5: Show linear independence
Assume:

$$\sum_{j=1}^{r} b_j f(v_j) = 0$$

Then:

$$f\left(\sum_{j=1}^{r} b_j (v_j)\right) = 0$$

So:

$$\sum_{j=1}^{r} b_j f(v_j) \in \ker(f)$$

But $\{u_1, \ldots, u_k, v_1, \ldots, v_r\}$ is a basis, so the vectors v_j are independent of the kernel basis. Hence:

$$b_j = 0 \qquad \forall j$$

So:
$\{f(v_1), \ldots, f(v_r)\}$ is linearly independent
Thus it is a basis of $Im(f)$
Step 6: Compute dimensions
We have:
• $\dim(\ker f) = k$
• $\dim(Imf) = r$
• $\dim(E) = k + r$
So:

$$\dim(Imf) + \dim(\ker f) = r + k = \dim(E)$$

or equivalently:

$$rank(f) + \dim(\ker f) = \dim(E)$$

Example 2.3.5

We have shown that $dim\ ker\ f_1 = 1$ for f_1 defined by
$f_1 : \mathbb{R}^2 \to \mathbb{R}$
$(x, y) \mapsto x + 2y$
Since $dim\ \mathbb{R}^2 = 2$ then, $dim\ Im\ (f_1) = dim\ \mathbb{R}^2 - dim\ ker\ f_1 = 2 - 1 = 1$.

Note 2.3.6

1. Always , the rank of f is either the same as or less than the dimension of E.
2. Additionally, if F has a finite dimension, we have $rg(f) \leq dim(F)$. Indeed, $Im(f)$ is a subspace of F, and therefore its dimension (equal to $rg(f)$ by definition) is less than or equal to that of F.
3. In the case when F has a finite dimension, then f is surjective if and only if $rg\ (f) = dim\ (F)$. Indeed, $Im(f)$ is a subspace of F, and therefore $Im(f) = F$ if and only if $dim(Im(f)) = dim(F)$.

Corollary 2.3.7

Assuming that E is of finite dimension. Let f be a linear mapping from E to F.

1. f is an injective linear application if and only if $rg(f) = dim(E)$. Then we have:
$$dim(E) \leq dim(F).$$
2. When f is surjective, then F is of finite dimension and
$$dim(F) \leq dim(E).$$
3. When f is bijective, then F is of finite dimension and
$$dim(F) = dim(E)$$

2.3.1 Rank theorem if $dim(E) = dim(F)$

A very important consequence of the rank theorem is the following:

Theorem 2.3.8

The following equivalencies hold if F has a finite dimension and $dim(E) = dim(F) = n$.

$$f \text{ injective} \Leftrightarrow f \text{ surjective} \Leftrightarrow f \text{ bijective} \Leftrightarrow f \text{ isomorphism}$$

This result applies in particular to endomorphisms.

Proof

We use the Rank– Theorem:

$$dim(Imf) + dim(kerf) = dim(E)$$

Step 1: Injective ⇒ Surjective
Assume f is injective.
Then:
$\ker(f) = \{0\} \Rightarrow \dim(kerf) = 0$
Using Rank–Theorem:

$$dim(E) = dim(Imf)$$

But:

$$dim(E) = dim(F) = n$$

So:

$$dim(Imf) = n = dim(F)$$

Since

$Im(f) \subseteq F$ and has the same dimension as F, we get:

$$Im(f) = F$$

Step 2: Surjective $\Rightarrow$ Injective

Assume f is surjective:

$$Im(f) = F \Rightarrow \dim(Imf) = \dim(F) = n$$

Using Rank–Theorem

$$n = dim(kerf) + dim(Imf)$$

So:

$$n = \dim(\ker f) + n \Rightarrow \dim(\ker f) = 0$$

Hence:

$$\ker(f) = \{0\}$$

So,
f is injective.
Step 3: Injective $\Leftrightarrow$ Surjective $\Rightarrow$ Bijective

A function is bijective if it is both injective and surjective.
From **Steps 1** and **2**:

$$\text{injective} \Leftrightarrow \text{surjective} \Rightarrow \text{bijective}$$

Step 4: Bijective $\Leftrightarrow$ Isomorphism
By definition:
A linear map is an isomorphism if it is:
• linear
• bijective
So:

$$\text{bijective} \Leftrightarrow \text{isomorphism}$$

Conclusion

$$f \text{ injective} \Leftrightarrow f \text{ surjective} \Leftrightarrow f \text{ bijective} \Leftrightarrow f \text{ isomorphism}$$

Example 2.3.9

1. The mapping f_1 is not an isomorphism because $dim\ \mathbb{R}^2 \neq dim\ \mathbb{R}$.

2. Let $g(x, y) = (2x - y, x - y)$, g defined from $\mathbb{R}^2$ to $\mathbb{R}^2$.
   We have g is an isomorphism ($\dim\mathbb{R}^2 = \dim \mathbb{R}^2 = 2$)
   because $dim\ ker\ g = 0$ indeed:

   $ker\ g = \{(x, y) \in \mathbb{R}^2, (2x - y, x - y) = (0, 0)\} = \{(0, 0)\}$,

   it is even an automorphism.

2.3.2 Composition of linear applications

Proposition 2.3.10

The composition of two linear mappings is still linear. More formally, it can be stated as:
$\forall p, q, r \in \mathbb{N}, \forall f \in Lq, r, \forall g \in Lp, q\ ,\ g \circ f$ is linear.

Note 2.3.11

Recall that Lp, q designates the set of linear mappings from $\mathbb{R}^q$ to $\mathbb{R}^p$.

$$\begin{aligned} \circ_{p,q,r} &: L_{p,q} \times L_{q,r} \to L_{p,r} \\ (g, f) &\mapsto g \circ f, \\ (g, f) &\mapsto (v \mapsto g(f(v))). \end{aligned}$$

It should be seen as follows:

$$\mathbb{R}^r \xrightarrow{f} \mathbb{R}^q \xrightarrow{g} \mathbb{R}^p$$

Proof

Let p, q, r be three integers, f in Lq, r and g in Lp, q. In order to demonstrate that $g \circ f$ is linear, we must:
$\forall \lambda, \mu \in \mathbb{R}, \forall u, v \in \mathbb{R}^r$, $(g \circ f)(\lambda u + \mu v) = \lambda (g \circ f)(u) + \mu (g \circ f)(v)$
We have: $(g \circ f)(\lambda u + \mu v) = g(f(\lambda u + \mu v))$ (by definition of composition)
$= g(\lambda f(u) + \mu f(v))$ (by linearity of f)
$= \lambda g(f(u)) + \mu g(f(v))$ (by linearity of g)

Example 2.3.12

Determine the composition $g \circ f$ using

$$\begin{aligned} g : \quad \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (x + y,\ x - y) \end{aligned}$$

$$\begin{aligned} f : \quad \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (3x + 5y + 7z,\ \ 2x + 2y + 2z) \end{aligned}$$

$$\begin{aligned} (g \circ f)(x, y, z) &= (g(f(x, y, z)), \text{ by using composition definition} \\ &= g(3x + 5y + 7z, 2x + 2y + 2z), \text{ by definition of } f \\ &= (5x + 7y + 9z, x + 3y + 5z), \text{ by definition of } g \end{aligned}$$

The composition $g \circ f$ is $(x, y, z) \mapsto (5x + 7y + 9z,\ \ x + 3y + 5z)$.

Proposition 2.3.13

The composition of linear applications is associative. More formally, this can be expressed as:
$\forall p, q, r, s \in \mathbb{N}, \forall h \in Lp, q, \forall g \in Lq, r, \forall f \in Lr, s, (h \circ g) \circ f = h \circ (g \circ f)$.

Proof

Let p, q, r, s, f , g , h ´We are going to prove
$(h \circ g) \circ f = h \circ (g \circ f)$ we say that
$\forall u \in \mathbb{R}^s$, $((h \circ g) \circ f)(u) = (h \circ (g \circ f))(u)$.
Let then u in $\mathbb{R}^s$. By usnig the definition of the composition we get

$$((h \circ g) \circ f)(u) = (h \circ g)(f(u)) = h(g(f(u))).$$

The same $(h \circ (g \circ f\ ))(u) = h((g \circ f\ )(u)) = h(g(f\ (u)))$.
Then, $((h \circ g) \circ f)(u) = (h \circ (g \circ f))(u)$.

2.3.3 Inverse of a Bijective Linear application, Automorphism.

Theorem 2.3.14

When f be an isomorphism from E to F, then, f^{-1} is an isomorphism from F to E.

Proposition 2.3.15

When f be an automorphism from E to F, then, f^{-1} is an automorphism from F to E.
Assume that E has two automorphisms: f and$g$. Then, $g \circ f$ is an automorphism of E, and we have:

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

2.4 Exercises

Exercise 2.4.1

Let the application f defied by $\mathbb{R}^2$ in $\mathbb{R}^2$ by:

$$f\ (x,y) = (x+y,\ x-y).$$

1. Prove that f is linear.
2. Find $ker\ f$, and $Im\ f$ and find the dimensions, f Is it bijective?
3. Determine $f \circ f$.

Exercise 2.4.2

Let f be the function defined from $\mathbb{R}^2$ to $\mathbb{R}^2$ by:

$$f(x,y) = (2x-4y,\quad x-2y).$$

1. Show that f is linear.
2. Determine $ker\ f$ and $Im\ f$, and give their dimensions. Is f bijective?

Exercise 2.4.3

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$f(x,y,z) = (x+y+z, -x+2y+2z).$$

Let $\beta = (e_1, e_2, e_3)$ be the canonical basis of $\mathbb{R}^3$ and $\beta' = (f_1, f_2)$ be the canonical basis of $\mathbb{R}^2$.

1. Show that f is a linear function.
2. Provide a basis and the dimension of $ker\ f$ and a basis and the dimension of $Im\ f$.

Exercise 2.4.4

Let f be the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$f\ \ (x_1,x_2,x_3) = (x_1 - x_3,\ 2x_1 + x_2 - 3x_3,\ \ -x_2 + 2x_3).$$

Let (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$.

1. Compute $f\ \ (e_1)$, $f\ \ (e_2)$, and $f\ \ (e_3)$.
2. Determine the coordinates of $f\ \ (e_1)$, $f\ \ (e_2)$, and $f\ \ (e_3)$ in the canonical basis.
3. Compute a basis for $ker\ f$ and a basis for $Im\ f$.

Exercise 2.4.5

Let f be the endomorphism of $\mathbb{R}^3$ whose image of the canonical basis $\beta = (e_1, e_2, e_3)$ is:

$$\begin{cases} f(e_1) = -7e_1 - 6e_2, \\ f(e_2) = 8e_1 + 7e_2, \\ f(e_3) = 6e_1 + 6e_2 - e_3. \end{cases}$$

1. For any vector $x = x_1e_1 + x_2e_2 + x_3e_3$, determine $f \circ f(x)$.
2. Deduce that f is invertible (i.e., bijective) and determine f^{-1}.

Exercise 2.4.6

1. Let $f_1 : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto x - y$ $\qquad$ $f_2 : \mathbb{R}^3 \to \mathbb{R}$ $(x, y, z) \mapsto y - z$

   and let $E = \{(x, y, z) \in \mathbb{R}^3 : f_1(x, y, z) = 0 \ \wedge f_2(x, y, z) = 0\}$

   a. Show that E is a subspace of $\mathbb{R}^3$.

   b. Give a basis and dimension of E

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$
   $(x, y, z) \mapsto (f_1(x, y, z),\ f_2(x, y, z),\ z)$

   a. Prove that f is a linear application

   b. Give a basis and the dimension of $ker\ f$

   c. Give a basis and the dimension of $Im\ f$

   d. Is f bijective?

2.5 Solution of exercises

Solution 2.4.1

1. f is linear if and only if

   $\forall \alpha, \beta \in \mathbb{R}, \forall (x,y),(x',y') \in \mathbb{R}^2; f(\alpha(x,y)+\beta(x',y')) \stackrel{?}{=} \alpha f(x,y)+\beta f(x',y')$.

   $$\begin{aligned} f(\alpha(x,y)+\beta(x',y')) &= f(\alpha x+\beta x', \alpha y+\beta y') \\ &= (\alpha x+\beta x'+\alpha y+\beta y', \alpha x+\beta x'-\alpha y-\beta y') \\ &= (\alpha x+\alpha y, \alpha x-\alpha y)+(\beta x'+\beta y', \beta x'-\beta y') \\ &= \alpha(x+y, x-y)+\beta(x'+y', x'-y') \\ &= \alpha f(x,y)+\beta f(x',y'). \end{aligned}$$

   Hence, f is linear.

2. Let's determine $ker f$ and $Im f$ and give their dimensions. Is f bijective?

   $ker f = \{(x,y) \in \mathbb{R}^2 \mid f(x,y)=(0,0)\} = \{(0,0)\}$, so $dim\, ker\, f = 0$.

   $Im f = \{(x+y, x-y) \mid (x,y) \in \mathbb{R}^2\} = \{x(1,1)+y(1,-1) \mid (x,y) \in \mathbb{R}^2\}$.

   Thus, $Im f$ is generated by two linearly independent vectors, so $dim\, Im\, f = 2$.

   Since the dimension of the domain equals the dimension of the codomain, f is bijective as it is both injective and surjective. Specifically, f is injective because $ker f = \{(0,0)\}$, and it is surjective because $dim\, \mathbb{R}^2 = dim\, Im\, f = 2$.

3. For any $(x,y) \in \mathbb{R}^2$, we have

   $f \circ f(x,y) = f(f(x,y)) = f(x+y, x-y) = (2x, 2y) = 2(x,y) = 2 \cdot \mathrm{Id}_{\mathbb{R}^2}$.

Solution 2.4.2

1. f is linear if and only if

   $\forall \alpha, \beta \in \mathbb{R}, \forall (x,y),(x',y') \in \mathbb{R}^2; f(\alpha(x,y)+\beta(x',y')) \stackrel{?}{=} \alpha f(x,y)+\beta f(x',y')$.

   $$\begin{aligned} f(\alpha(x,y)+\beta(x',y')) &= f(\alpha x+\beta x', \alpha y+\beta y') \\ &= (2\alpha x+2\beta x'-4\alpha y-4\beta y', \alpha x+\beta x'-2\alpha y-2\beta y') \\ &= (2\alpha x-4\alpha y, \alpha x-2\alpha y)+(2\beta x'-4\beta y', \beta x'-2\beta y') \\ &= \alpha f(x,y)+\beta f(x',y'). \end{aligned}$$

   Thus, f is linear.

2. Let's determine $ker f$ and $Im f$ and give their dimensions. Is f bijective?

   $ker f = \{(x,y) \in \mathbb{R}^2 \mid f(x,y)=(0,0)\} = \{(2y,y) \mid y \in \mathbb{R}\} = \{y(2,1) \mid y \in \mathbb{R}\}$.

   So, $dim\, ker\, f = 1$. Therefore, f is not injective.

   $Im f = \{(2x-4y, x-2y) \mid (x,y) \in \mathbb{R}^2\} = \{x(2,1)+y(-4,-2) \mid (x,y) \in \mathbb{R}^2\}$.

   $Im f$ is generated by two dependent vectors since $(-4,-2) = -2(2,1)$. Hence, $dim\, Im\, f = 1$. Alternatively, we can use the fact that $dim\, ker\, f + dim\, Im\, f = dim\, \mathbb{R}^2 \Rightarrow dim\, Im\, f = 2-1 = 1$.

3. f is not bijective as it is neither injective nor surjective.

Solution 2.4.3

1. Let $u=(x,y,z)$ and $u'=(x',y',z')$ be two vectors in $\mathbb{R}^3$, and let λ and λ' be real numbers.

   $\lambda u+\lambda' u' = (\lambda x+\lambda' x', \lambda y+\lambda' y', \lambda z+\lambda' z')$

   $f(\lambda u+\lambda' u') = (\lambda x+\lambda' x'+\lambda y+\lambda' y'+\lambda z+\lambda' z', -(\lambda x+\lambda' x')+2(\lambda y+\lambda' y')+2(\lambda z+\lambda' z'))$

$$= (\lambda(x+y+z)+\lambda'(x'+y'+z'), \lambda(-x+2y+2z)+\lambda'(-x'+2y'+2z'))$$
$$= \lambda(x+y+z,-x+2y+2z)+\lambda'(x'+y'+z',-x'+2y'+2z')$$
$$= \lambda f(u)+\lambda' f(u')$$

Therefore, f is linear.

2. $\ker f = \{(x,y,z)\in\mathbb{R}^3 \mid f(x,y,z)=(0,0)\}$
$= \{(x,y,z)\in\mathbb{R}^3 \mid (x+y+z,-x+2y+2z)=(0,0)\}$
$\Rightarrow \begin{cases} x+y+z=0 \\ 3y+3z=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=-z \end{cases}$
$\ker f = \{(0,-z,z) \mid z\in\mathbb{R}\}$
$= \{z(0,-1,1) \mid z\in\mathbb{R}\}$

Let $f=(0,-1,1)$; f is a basis for $\ker(f)$.

$Imf = \{f(e_1), f(e_2), f(e_3)\}$

$$\begin{cases} f(e_1)=(1,-1)=f_1-f_2 \\ f(e_2)=(1,2)=f_1+2f_2 \\ f(e_3)=(1,2)=f_1+2f_2 \end{cases}$$

$Imf = \{f_1-f_2, f_1+2f_2\}$

f_1-f_2 and f_1+2f_2 are not proportional, so they form a linearly independent set in Imf. As they generate Imf, they form a basis for Imf, and thus $dim(Imf)=2$. Note that $Imf=\mathbb{R}^2$.

Solution 2.4.4

1. $$\begin{cases} f(e_1)=(1,2,0)=1\cdot e_1+2\cdot e_2+0\cdot e_3 \\ f(e_2)=(0,1,-1)=0\cdot e_1+1\cdot e_2+(-1)\cdot e_3 \\ f(e_3)=(-1,-3,2)=(-1)\cdot e_1+(-3)\cdot e_2+2\cdot e_3 \end{cases}$$

2. - Coordinates of $f(e_1)$ in the basis (e_1,e_2,e_3) are $(1,2,0)$.
   - Coordinates of $f(e_2)$ in the basis (e_1,e_2,e_3) are $(0,1,-1)$.
   - Coordinates of $f(e_3)$ in the basis (e_1,e_2,e_3) are $(-1,-3,2)$.

3. $(x,y,z)\in\ker f \Leftrightarrow \begin{cases} x-z=0 \\ 2x+y-3z=0 \\ -y+3z=0 \end{cases} \Leftrightarrow \begin{cases} x-z=0 \\ y-z=0 \\ -y+3z=0 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \\ z=0 \end{cases}$

Therefore, $\ker f = \{0_{\mathbb{R}^3}\}$.

First method:

$Imf = \{f(e_1), f(e_2), f(e_3)\}$

Check if the family $\{f(e_1), f(e_2), f(e_3)\}$ is linearly independent:

$\alpha f(e_1)+\beta f(e_2)+\gamma f(e_3)=0_{\mathbb{R}^3} \Leftrightarrow \alpha\cdot(1,2,0)+\beta\cdot(0,1,-1)+\gamma\cdot(-1,-3,2)=(0,0,0)$

$\Rightarrow \begin{cases} \alpha-\gamma=0 \\ 2\alpha+\beta-3\gamma=0 \\ -\beta+2\gamma=0 \end{cases}$

This is the same system as before, so $\alpha=\beta=\gamma=0$. The family is linearly independent and spans Imf, so it forms a basis for Imf, and $dim(Imf)=3$, concluding that $Imf=\mathbb{R}^3$.

Second method (more complicated):

$$Imf = \{f(e_1), f(e_2), f(e_3)\} = \{e_1+2e_2, e_2-e_3, -e_1-3e_2+2e_3\}$$
$$= \{e_1+2e_2, e_2-e_3, -e_2+2e_3+e_1+2e_2\}$$
$$= \{e_1+2e_2, e_2-e_3-e_2+2e_3, -e_2+2e_3\}$$
$$= \{e_1+2e_2, e_3, -e_2+2e_3\}$$
$$= \{e_1+2e_2, e_3, -e_2+2e_3-2e_3\}$$

$$= \{e_1 + 2e_2, e_3, -e_2\}$$
$$= \{e_1 + 2e_2, e_3, e_2\}$$
$$= \{e_1, e_3, e_2\}$$
$$= \{e_1, e_2, e_3\}$$

Thus, a basis for Imf is (e_1, e_2, e_3), and of course, $Imf = \mathbb{R}^3$.

Third method:

Using the rank-nullity theorem, $dim(\ker f) + dim(Imf) = dim(\mathbb{R}^3) = 3$. Since $dim(\ker f) = 0$, $dim(Imf) = 3$, and therefore $Imf = \mathbb{R}^3$. A basis for Imf is $(f(e_1), f(e_2), f(e_3))$.

Solution 2.4.5

1. The matrix of $f \circ f$ in the basis β is $\mathcal{M}_\beta(f) \times \mathcal{M}_\beta(f)$, where

$$\mathcal{M}_\beta(f) = \begin{pmatrix} -7 & 8 & 6 \\ -6 & 7 & 6 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore, $\mathcal{M}_\beta(f) \times \mathcal{M}_\beta(f) = \begin{pmatrix} -7 & 8 & 6 \\ -6 & 7 & 6 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} -7 & 8 & 6 \\ -6 & 7 & 6 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Id_{\mathbb{R}^3}$.

2. There exists g such that $g \circ f = Id$, so f is bijective, and $f^{-1} = g = f$.

Solution ??

1. Let

$$f_1 : \mathbb{R}^3 \to \mathbb{R} \quad , \qquad f_2 : \mathbb{R}^3 \to \mathbb{R} \qquad (x, y, z) \mapsto x - y$$
$(x, y, z) \mapsto y - z$

and let $E = \left\{(x, y, z) \in \mathbb{R}^3 : f_1(x, y, z) = 0 \ \wedge f_2(x, y, z) = 0\right\}$.

a. Show that E is a subspace of $\mathbb{R}^3$ Since

$$f1(x, y, z) = 0 \Leftrightarrow x = y,$$

and

$$f_2(x, y, z) = 0 \Leftrightarrow y = z,$$

we obtain

$$x = y = z.$$

Hence,

$$E = \{(t, t, t), \quad t \in \mathbb{R}\}.$$

So,

$$E = \{t(1, 1, 1), \quad t \in \mathbb{R}\}.$$

To prove that E is a vector subspace:

• The zero vector belongs to E

$$(0, 0, 0) = (0)(1, 1, 1) \in E.$$

• Let $u = (t, t, t)$ and $v = (s, s, s)$ be in E. Then

$$u + v = (t + s, t + s, t + s) \in E.$$

• Let $\lambda \in \mathbb{R}$ and $u = (t, t, t) \in E$. Then

$$\lambda u = (\lambda t, \ \lambda t, \ \lambda t) \in E$$

Therefore

E is a vector subspace of $\mathbb{R}^3$

b. Give a basis and the dimension of E

Since

$$E = \{(t,t,t) : t \in \mathbb{R}\} = \{t(1,1,1) : t \in \mathbb{R}\},$$

we have

$$E = Span\{(1,1,1)\}$$

Thus a basis of E is

$$\{(1,1,1)\}$$

and

$$\dim(E) = 1$$

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x,y,z) = (f_1(x,y,z),\ f_2(x,y,z),\ z) = (x\ -y,\ y\ -z,\ z).$$

a. Prove that f is linear

Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$

Then

$f(u+v) = f(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$

$= ((x1 + x2) - (y1 + y2)\ , (y_1 + y_2) - (z_1 + z_2),\ z_1 + z_2)$

$= f(u) + f(v).$

Also, for $\lambda \in \mathbb{R}$,

$f(\lambda u) = f(\lambda x_1, \lambda y_1, \lambda z_1)$

$= (\lambda(x_1 - y_1), \lambda(y_1 - z_1), \lambda z_1)$

$= \lambda f\ (u).$

Hence f is linear.

$$f \text{ is a linear application}$$

b. Give a basis and the dimension of $\ker(f)$

The kernel is

$$\ker\ (f\ ) = \{(x,y,z) \in \mathbb{R}^3 : f\ (x,y,z) = (0,0,0)\}.$$

Thus,

$$\begin{cases} x - y = 0, \\ y - z = 0, \\ z = 0 \end{cases}$$

From $z = 0$, we get $y = 0$, and then $x = 0$.

Therefore

$$\ker\ (f\ ) = \{(0,0,0)\}.$$

Hence

$$\dim(\ker(f\ )) = 0.$$

A basis of the zero subspace is the empty set ϕ

c. Give a basis and the dimension of $Im\ (f)$

Compute the images of the canonical basis vectors:

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1).$$

$f(e_1) = (1,0,0)$,
$f(e_2) = (-1,1,0)$,
$f(e_3) = (0,-1,1)$.
Therefore

$$Im(f) = \text{Span}\left\{(1,0,0), \quad (-1,1,0), \quad (0,-1,1)\right\}.$$

Let

$$a(1,0,0) + b(-1,1,0) + c(0,-1,1) = (0,0,0).$$

This gives

$$(a-b, b-c, c) = (0,0,0).$$

Hence

$$c = 0, \quad b = 0, \quad a = 0.$$

The three vectors are linearly independent.
Thus they form a basis of the image.
So,

$$\dim\ (Im\ (f\ )) = 3$$

Since $Im(f)$ is a 3-dimensional subspace of $\mathbb{R}^3$,

$$Im(f\ ) = \mathbb{R}^3$$

d. Is f bijective?
Using the rank-nullity theorem

$$\dim(\ker f\ ) + \dim\ (Im\ f\ ) = 3.$$

We found

$$\dim(\ker f\ ) = 0, \qquad \dim\ (Im\ f\ ) = 3.$$

Therefore
f is injective and surjective.
Hence,

$$f \text{ is bijective}$$

Chapter 3

Matrices

3.1 Introduction

This chapter is devoted to the study of matrices, which constitute a fundamental tool in linear algebra for representing and manipulating linear transformations in a concrete and computational form. Matrices provide a bridge between abstract linear applications and explicit calculations, allowing vector space transformations to be expressed in terms of arrays of scalars.

The concept of a matrix associated with a linear map plays a central role in this chapter, as it enables the translation of theoretical problems into algebraic computations. In this context, matrices are not only computational tools but also algebraic objects with rich structure, forming a vector space and, in the case of square matrices, a non-commutative ring.

Special attention is given to matrix operations, invertible matrices, and the determinant, which is a key invariant used to study invertibility and structural properties. The notion of rank is also revisited in the context of matrices, highlighting its relationship with linear transformations and its invariance under transposition.

Furthermore, this chapter explores the relationship between linear maps and their associated matrices, as well as the effect of change of basis, which is essential for simplifying matrix representations. Finally, an introduction to diagonalization is presented, providing a powerful tool for analyzing square matrices and simplifying their structure in suitable bases.

This chapter aims to develop both theoretical understanding and computational skills in matrix theory, which are essential for further studies in linear algebra and its applications.

3.2 Matrix Associated with a Linear Map

Let $\mathbb{K}$ be a commutative field.
Let E and F be two $\mathbb{K}$ vector spaces of finite dimension n and m, f a linear map from E to F. Let $B = \{e_1, e_2, ..., e_n\}$ be a basis of E, $B^{'} = \{e_1^{'}, e_2^{'}, ..., e_m^{'}\}$ be a basis of F. Since $f(e_1), f(e_2), ..., f(e_n)$ are vectors in F and $\{e_1^{'}, e_2^{'}, ..., e_m^{'}\}$ is a basis of F, then $f(e_1)$, $f(e_2)$, ..., $f(e_n)$ can be written as linear combinations of the vectors in the basis $B^{'} = \{e_1^{'}, e_2^{'}, ..., e_m^{'}\}$. For every $j = 1, ..., n$, we have:

$$f(e_j) = a_{1j}e_1^{'} + a_{2j}e_2^{'} + ... + a_{mj}e_m^{'} = \sum_{j=1}^{n} a_{ij}e_i^{'}, \qquad i = 1, ..., m$$

Then, we have $(f(e_1), f(e_2), ..., f(e_n)) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} e_1^{'} \\ e_2^{'} \\ \vdots \\ e_m^{'} \end{bmatrix}$

and

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called the matrix associated with f relative to the basis B and $B^{'}$. The matrix is denoted by (a_{ij}) where i denotes the row index and j denotes the column index.

Now, let's introduce the concept of matrices and algebraic operations on matrices.

Definition 3.2.1

A matrix in $\mathbb{K}$ of type (n,p) is a rectangular array A of elements from IK with n rows and p columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}.$$

We denote a_{ij} as the element at row number i and column j, and we represent the matrix A by $A = (a_{ij})_{1\leq i\leq n,\ 1\leq j\leq p}$. The set of matrices of type (n,p) is denoted as $\mathcal{M}_{(n,p)}(IK)$.

1. For $n=1$, we say that A is a row matrix, $A = (a_{11}, a_{12}, ..., a_{1p})$.
2. For $p=1$, we say that A is a column matrix, $A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$
3. For $n=p$, we say that A is a square matrix of order n, and we denote $A \in \mathcal{M}_n(\mathbb{K})$.

Example 3.2.2

1. $A_1 = \begin{pmatrix} -10 & 2 & 0 \\ 1 & -4 & 9 \\ -5 & -7 & 0 \\ -3 & -1 & 0 \end{pmatrix}$, A_1 is a matrix of type $(4,3)$.
2. $A_2 = \begin{pmatrix} -4 & -1 & 3 \\ 0 & -2 & -7 \end{pmatrix}$, A_2 is a matrix of type $(2,3)$.
3. $A_3 = \begin{pmatrix} -1 & -5 \\ 4 & -4 \end{pmatrix}$, A_3 is a square matrix of order 2.

3.3 Vector Space of Matrices with n Rows and m Columns

3.3.1 Matrix Operations

Definition 3.3.1

Let $A = (a_{ij})_{1\leq i\leq n,\ 1\leq j\leq p}$ and $B = (b_{ij})_{1\leq i\leq n,\ 1\leq j\leq p}$ be two matrices of types (n,p),

1. We say that $A = B$ if $\forall i = 1, ..., n$, $\forall j = 1, ..., p$; $a_{ij} = b_{ij}$.
2. The transpose of matrix A is a matrix denoted by A^t defined by:

$$A^t = (a_{ji})_{1\leq j\leq p,\ 1\leq i\leq n}.$$

In other words, A^t is the matrix of type (p,n) obtained by replacing the rows with the columns and the columns with the rows, and we have:

$$(A^t)^t = A.$$

Example 3.3.2

1. $A_1 = \begin{pmatrix} 1 & 0 \\ -4 & 6 \\ -3 & -5 \end{pmatrix} \Rightarrow A_1^t = \begin{pmatrix} 1 & -4 & -3 \\ 0 & 6 & -5 \end{pmatrix}$.

2. $A_2 = \begin{pmatrix} 5 & 7 & 1 & 0 & -10 \\ -8 & 0 & 5 & -13 & 5 \\ 7 & 9 & 3 & -2 & 1 \\ -1 & 0 & 5 & 0 & 0 \end{pmatrix} \Rightarrow A_2^t = \begin{pmatrix} 5 & -8 & 7 & -1 \\ 7 & 0 & 9 & 0 \\ 1 & 5 & 3 & 5 \\ 0 & -13 & -2 & 0 \\ -10 & 5 & 1 & 0 \end{pmatrix}$.

3. $A_3 = \begin{pmatrix} 0 & -1 \\ 5 & -8 \end{pmatrix} \Rightarrow A_3^t = \begin{pmatrix} 0 & 5 \\ -1 & -8 \end{pmatrix}$.

Sum of Matrices

Theorem 3.3.3

By equipping the set $\mathcal{M}_{(n,p)}(\mathbb{K})$ with the following operations:

$$(+) : \mathcal{M}_{(n,p)}(\mathbb{K}) \times \mathcal{M}_{(n,p)}(\mathbb{K}) \to \mathcal{M}_{(n,p)}(\mathbb{K})$$

$$\left(\begin{pmatrix} a_{11} & a_{12} & .. & a_{1p} \\ a_{21} & a_{22} & .. & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & .. & a_{np} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & .. & b_{1p} \\ b_{21} & b_{22} & .. & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & .. & b_{np} \end{pmatrix} \right) \mapsto \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & .. & a_{1p}+b_{1p} \\ a_{21}+b_{21} & a_{22}+b_{22} & .. & a_{2p}+b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & .. & a_{np}+b_{np} \end{pmatrix},$$

and

$$(\times) : \quad \mathbb{K} \times \mathcal{M}_{(n,p)}(\mathbb{K}) \quad \to \mathcal{M}_{(n,p)}(\mathbb{K}).$$

$$\left(\lambda, \begin{pmatrix} a_{11} & a_{12} & .. & a_{1p} \\ a_{21} & a_{22} & .. & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & .. & a_{np} \end{pmatrix} \right) \mapsto \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & .. & \lambda a_{1p} \\ \lambda a_{21} & \lambda a_{22} & .. & \lambda a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & .. & \lambda a_{np} \end{pmatrix}.$$

Then $(\mathcal{M}_{(n,p)}(\mathbb{K}), +, \cdot)$ is an $\mathbb{K}$-vector space of dimension $n \times p$,

where the additive identity is the zero matrix $\begin{pmatrix} 0 & 0 & .. & 0 \\ 0 & 0 & .. & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & .. & 0 \end{pmatrix}$..

Product of Two Matrices

Definition 3.3.4

Let $A \in \mathcal{M}_{(n,p)}(\mathbb{K})$ and $B \in \mathcal{M}_{(p,m)}(\mathbb{K})$, the product of matrix A by B is defined as a matrix $C = (c_{ij})_{1 \leq i \leq n,\ 1 \leq j \leq m} \in \mathcal{M}(n,m)(\mathbb{K})$, with

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + ... + a_{ip}b_{pj}.$$

Note 3.3.5

1. The element c_{ij} of matrix C is calculated by adding the product of the elements in the i-th row of matrix A by the elements in the j-th column of matrix B.
2. The product of two matrices is possible only if the number of columns in matrix A is equal to the number of rows in matrix B.

Example 3.3.6

$$A = \begin{pmatrix} 1 & 0 & 9 \\ 4 & 7 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 5 & 0 & 2 & 1 \\ 0 & 3 & 0 & 0 \end{pmatrix}$$

A is of type $(2,3)$ and B is of type $(3,4)$, so C will be of type $(2,4)$.

$$C = A{\cdot}B = \begin{pmatrix} 1.2+0.5+9.0 & 1.1+0.(0)+9.3 & 1.(3)+0.(2)+9.0 & 1.(1)+0.(1)+9.0 \\ 4.2+7.5+(-4).0 & 4.1+7.(0)+(-4).3 & 4.(3)+7.(2)+(-4).0 & 4.(1)+7.(1)+(-4).0 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 28 & 3 & 1 \\ 43 & -8 & 26 & 11 \end{pmatrix}$$

Note 3.3.7

The product of two matrices is not commutative, here is an example:

$$A \times B = \begin{pmatrix} 1 & 2 \\ -4 & 5 \end{pmatrix} \times \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 10 & -17 \end{pmatrix}$$

$$B \times A = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} -12 & 15 \\ 6 & -1 \end{pmatrix}$$

So $A \times B \neq B \times A$

3.4 Square Matrix Ring

Definition 3.4.1

Let A be a square matrix of order n, $A = (a_{ij})_{1 \leq i \leq n,\ 1 \leq j \leq n}$,

1. The sequence of elements $\{a_{11}, a_{22}, ..., a_{nn}\}$ is called the principal diagonal of A.
2. The trace of A is the number
$$Tr(A) = a_{11} + a_{22} + ... + a_{nn}.$$
3. A is called a diagonal matrix if $a_{ij} = 0$, $\forall i \neq j$, meaning all elements of A are zero except on the principal diagonal.
4. A is called an upper (resp. lower) triangular matrix if $a_{ij} = 0$, $\forall i > j$ (resp. $i < j$), meaning elements below (resp. above) the diagonal are zero.
5. A is called symmetric if $A = A^t$.

Example 3.4.2

1. $A_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, A_1 is a diagonal matrix.

2. $A_2 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -14 & 0 \\ -3 & 7 & 2 \end{pmatrix}$, A_2 is a lower triangular matrix.

3. $A_3 = \begin{pmatrix} 2 & 10 & -25 \\ 0 & -1 & 22 \\ 0 & 0 & 1 \end{pmatrix}$, A_3 is an upper triangular matrix.

4. $A_4 = \begin{pmatrix} 2 & 10 & -2 \\ 10 & -1 & 21 \\ -2 & 21 & 1 \end{pmatrix} \Rightarrow A_4^t = \begin{pmatrix} 2 & 10 & -2 \\ 10 & -1 & 21 \\ -2 & 21 & 1 \end{pmatrix}$, A_4 is a symmetric matrix.

Proposition 3.4.3

The matrix product is an internal operation in $\mathcal{M}_{(n,n)}(IK)$, and it has a neutral element called the identity matrix, denoted by I_n, defined as:

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

3.4.1 Invertible Matrices

Definition 3.4.4

Let $A \in \mathcal{M}_{(n,n)}(IK)$, we say that A is invertible if there exists a matrix $B \in \mathcal{M}_{(n,n)}(IK)$ such that:

$$A.B = B.A = I_n.$$

Example 3.4.5

Show that the matrix $A = \begin{pmatrix} 1 & 1 \\ 4 & -3 \end{pmatrix}$ is invertible.

By seeking the matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that:

$$A.B = \begin{pmatrix} 1 & 1 \\ 4 & -3 \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 4 & -3 \end{pmatrix} = B.A$$

This gives:

$$A.B = \begin{pmatrix} a+c & b+d \\ 4a-3c & 4b-3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+4b & a-3b \\ c+4d & c-3d \end{pmatrix} = B.A$$

Which leads to the system:

$$\Rightarrow \begin{cases} a+c=1 \\ 4a-3c=0 \\ b+d=0 \\ 4b-3d=1 \end{cases} \Rightarrow \begin{cases} a=\frac{3}{7} \\ c=\frac{4}{7} \\ b=\frac{1}{7} \\ d=\frac{-1}{7} \end{cases}$$

So, $B = \begin{pmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{pmatrix}$. Then, $B = \frac{1}{7}.\begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}$

3.4.2 Determinant of a Square Matrix and Properties

Definition 3.4.6

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, be a matrix in $\mathcal{M}_{(2,2)}(\mathbb{R})$, the determinant of A is the real number given by:

$$det(A) = a_{11} \times a_{22} - a_{12}a_{21}$$

It is denoted as $det(A)$ or $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

Example 3.4.7

Calculate $det(A)$ for $A = \begin{pmatrix} 1 & -4 \\ 2 & 5 \end{pmatrix}$.

$$det(A) = |A| = \begin{vmatrix} 1 & -4 \\ 2 & 5 \end{vmatrix} = 1 \times 5 - 2 \times (-4) = 13$$

Definition 3.4.8

Similarly, the determinant of a 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathcal{M}_{(3,3)}(\mathbb{R})$ is given by:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Where C_{11}, C_{12}, and C_{13} are the cofactors of the elements a_{11}, a_{12}, and a_{13}, respectively.

Example 3.4.9

Calculate $|A|$ for $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 2 & 1 & -1 \end{pmatrix}$.

$$|A| = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= 1(-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} + 0(-1)^{1+3}\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= 1(-3-1) - 2(1-2) + 0(-1-6) = -2.$$

Proposition 3.4.10

To calculate the determinant of a matrix A, one can expand A along any row or column. Following this proposition, it is better to choose the row or column containing the most zeros.

Example 3.4.11

Using the same matrix as the previous example.
Method 1: calculating along the third row, we have:

$$det(A) = |A| = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 2 & 1 & -1 \end{vmatrix} = (-1)^{3+1}(2)\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + (-1)^{3+2}(+1)\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} + (-1)^{3+3}(-1)\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

Therefore:

$$det(A) = |A| = 2(2) - 1 - 5 = 4 - 6 = -2.$$

Method 2: Calculate $|A|$ for

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

.

$$det(A) = |A| = a_{11}(a_{22})(a_{23}) + a_{12}(a_{23})(a_{31}) + a_{21}(a_{32})(a_{13}) - [a_{31}(a_{22})(a_{13}) + a_{32}(a_{23})(a_{11}) + a_{21}(a_{12})(a_{33})]$$

$$= 1(3)(-1) + 2(1)(2) + 1(-1)(0) - [2(3)(0) + 1(1)(1) + 2(-1)(-1)] = 1 - 3 = -2$$

Definition 3.4.12

Similarly, the determinant of a 4×4 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in \mathcal{M}_{(4,4)}(\mathbb{R})$ is given by:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = (-1)^{1+1}(a_{11})\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} + \ldots + (-1)^{1+4}(a_{14})\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Definition 3.4.13

Let $A = (a_{ij})_{1 \le i \le n, 1 \le j \le n}$ be a matrix. The determinant along the j-th column is given by:

$$\det(A) = (-1)^{1+j}a_{1j}D_{1j} + (-1)^{2+j}a_{2j}D_{2j} + \cdots + (-1)^{n+j}a_{nj}D_{nj}, \quad j = 1, \ldots, n.$$

The determinant along the i-th row is given by:

$$\det(A) = (-1)^{i+1}a_{i1}D_{i1} + (-1)^{i+2}a_{i2}D_{i2} + \cdots + (-1)^{i+n}a_{in}D_{in}, \quad i = 1, \ldots, n.$$

Here, A_{ij} represents the minor determinant of the term a_{ij}, which is the determinant of order $n-1$ obtained from $\det(A)$ by removing the i-th row and j-th column.

Proposition 3.4.14

Let $A \in \mathcal{M}_n(IK)$. We have:

1. $\det(A) = \det(A^t)$.
2. $\det(A) = 0$ if two rows (or two columns) of A are equal.
3. $\det(A) = 0$ if two rows (or two columns) of A are proportional.
4. $\det(A) = 0$ if one row is a linear combination of two other rows of A (similarly for columns).

5. $\det(A)$ remains unchanged if a linear combination of other rows is added to one row (similarly for columns).
6. If $B \in \mathcal{M}_n(IK)$, then $\det(A \cdot B) = \det(A) \cdot \det(B)$.

Example 3.4.15

1. $|A| = \begin{vmatrix} 1 & 0 & -5 \\ 5 & 4 & -2 \\ 1 & 0 & -5 \end{vmatrix} = 0$, because row 1 is equal to row 3, $L_1 = L_3$.

2. $|B| = \begin{vmatrix} 2 & -4 & 6 & 10 \\ 0 & 8 & -7 & 1 \\ 1 & -2 & 3 & 5 \\ 2 & -3 & 0 & -1 \end{vmatrix} = 0$, because $L_1 = 2L_3$.

3. $|C| = \begin{vmatrix} 2 & -3 & 2 & 6 \\ 1 & 8 & 1 & -1 \\ 0 & -4 & 0 & 5 \\ -2 & -3 & -2 & -1 \end{vmatrix} = 0$, because $C_1 = C_3$.

Definition 3.4.16

Let $V_1, V_2, \ldots, V_n$ be n vectors in $\mathbb{R}^n$. The determinant of the vectors $(V_1, V_2, \ldots, V_n)$, denoted as

$$\det(V_1, V_2, \ldots, V_n)$$

, is the determinant whose columns are the vectors $V_1, V_2, \ldots, V_n$.

Example 3.4.17

Let $V_1 = (-1, -1, 0)$, $V_2 = (0, -1, 0)$, $V_3 = (0, 1, 1)$. Then

$$\begin{aligned} \det(V_1, V_2, V_3) &= \begin{vmatrix} -1 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \\ &= (-1)\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} \\ &= (-1)(-1) \\ &= 1. \end{aligned}$$

Proposition 3.4.18

The vectors $(V_1, V_2, \ldots, V_n)$ form a basis for $\mathbb{R}^n$ if and only if $\det(V_1, V_2, \ldots, V_n) \neq 0$.

Example 3.4.19

Let $V_1 = (-1, -2, 0)$, $V_2 = (0, -1, -1)$, $V_3 = (0, 2, 1)$. They form a basis for $\mathbb{R}^3$ because:

$$\det(V_1, V_2, V_3) \neq 0.$$

3.4.3 Rank of a Matrix (Associated Application)

Definition 3.4.20

Let $A \in \mathcal{M}(n,p)(IK)$. The rank of A, denoted as rgA, is the order of the largest square matrix B extracted from A such that $\det(B) \neq 0$.

Example 3.4.21

1. $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \Rightarrow \det(A) = 1 \times 4 - (-1) \times 2 = 6 \neq 0 \Rightarrow rgA = 2.$

2. $B = \begin{pmatrix} 1 & -4 \\ -1 & 4 \end{pmatrix} \Rightarrow \det(B) = 0$; thus $rgB = 1$.

3. $C = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & -1 \end{pmatrix}$, $rgC < 4$ ($rgC \leq 3$). The largest square matrix contained in C is of order 3. In this example, there are 4 possibilities:

$$C_1 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

$\det(C_1) = \det(C_2) = \det(C_3) = \det(C_4) = 0$, so $rgC < 3$, and we have:

$$\begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = -1 \neq 0 \Rightarrow rgC = 2.$$

Theorem 3.4.22

The rank of a matrix is equal to the maximum number of linearly independent row (or column) vectors.

Definition 3.4.23

Let $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq n} \in \mathcal{M}_n(IK)$. The cofactor of index i and j of A is the scalar $c_{ij} = (-1)^{i+j} \det A_{ij}$, where A_{ij} is the matrix obtained from A by removing the i-th row and the j-th column. The matrix $C = (c_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ is called the matrix of cofactors, and the transpose of C is called the adjugate or comatrix of A.

Example 3.4.24

Consider the matrix $A = \begin{pmatrix} 1 & -2 & 0 \\ 5 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$.

The cofactors of A are calculated as follows:

$$c_{11} = (-1)^{1+1}\det(A_{11}) = (-1)^2 \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = -4,$$

$$c_{12} = (-1)^{1+2}\det(A_{12}) = (-1)^3 \begin{vmatrix} 5 & 1 \\ 0 & -1 \end{vmatrix} = 5,$$

$$c_{13} = (-1)^{1+3}\det(A_{13}) = (-1)^4 \begin{vmatrix} 5 & 3 \\ 0 & 1 \end{vmatrix} = 5,$$

$$c_{21} = (-1)^{2+1}\det(A_{21}) = (-1)^3 \begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} = -2,$$

$$c_{22} = (-1)^{2+2}\det(A_{22}) = (-1)^4 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1,$$

$$c_{23} = (-1)^{2+3}\det(A_{23}) = (-1)^5 \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1,$$

$$c_{31} = (-1)^{3+1}\det(A_{31}) = (-1)^4 \begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix} = -2,$$

$$c_{32} = (-1)^{3+2}\det(A_{32}) = (-1)^5 \begin{vmatrix} 1 & 0 \\ 5 & 1 \end{vmatrix} = -1,$$

$$c_{33} = (-1)^{3+3}\det(A_{33}) = (-1)^6 \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix} = 13.$$

The matrix of cofactors is:

$$C = \begin{pmatrix} -4 & 5 & 5 \\ -2 & -1 & 1 \\ -2 & -1 & 13 \end{pmatrix}$$

The transpose of the matrix of cofactors (adjugate or comatrix of A) is:

$$C^{\,t} = \begin{pmatrix} -4 & -2 & -2 \\ 5 & -1 & -1 \\ 5 & 1 & 13 \end{pmatrix}$$

Theorem 3.4.25

Let $A \in \mathcal{M}_n(IK)$, then:

$$A \text{ is invertible} \Leftrightarrow \det(A) \neq 0$$

In this case, the inverse of matrix A is given by:

$$A^{-1} = \frac{1}{\det(A)} C^{\,t}$$

where $C^{\,t}$ is the adjugate (or comatrix) of A.

Example 3.4.26

Consider the matrix $A = \begin{pmatrix} 1 & -2 & 0 \\ 5 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$.

Here, $\det(A) = -14 \neq 0$, so A is invertible. The inverse of A is calculated as:

$$A^{-1} = \frac{1}{\det(A)} C^{\,t} = \frac{1}{-14} \begin{pmatrix} -4 & -2 & -2 \\ 5 & -1 & -1 \\ 5 & 1 & 13 \end{pmatrix} = \begin{pmatrix} \frac{4}{14} & \frac{2}{14} & \frac{2}{14} \\ \frac{-5}{14} & \frac{1}{14} & \frac{1}{14} \\ \frac{-5}{14} & \frac{1}{14} & \frac{-13}{14} \end{pmatrix}$$

One can verify that:

$$A^{-1}A = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = AA^{-1}.$$

3.5 Relations between a linear map and its associated matrix

Definition 3.5.1

Let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be the matrix of f with respect to the basis B and B'. If $E = F$ and $B = B'$, we say that A is the matrix of f with respect to the base B and denote it by $\mathcal{M}_{(B)}(f)$.

Example 3.5.2

Consider the linear map:

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto f(x, y, z) = (x + y + z, x - y). \end{aligned}$$

Using the canonical basis $B = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ for $\mathbb{R}^3$ and $B' = \{v_1 = (1, 0), v_2 = (0, 1)\}$ for $\mathbb{R}^2$, we have:

$$\begin{aligned} f(x, y, z) &= f(xe_1 + ye_2 + ze_3) \\ &= f(xe_1) + f(ye_2) + f(ze_3) \\ &= f(x(1, 0, 0)) + f(y(0, 1, 0)) + f(z(0, 0, 1)) \\ &= f((x, 0, 0)) + f((0, y, 0)) + f((0, 0, z)) \\ &= (x, x) + (y, -y) + (z, 0) \\ &= x(1, 1) + y(1, -1) + z(1, 0) \end{aligned}$$

So,

$$\mathcal{M}_{(B,B')}(f) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Example 3.5.3

Consider the linear map:

$$\begin{aligned} g : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto f(x, y) = (x + y, x - y). \end{aligned}$$

With basis $B = \{e_1 = (1, 2), e_2 = (-1, 1)\}$ and $B' = \{v_1 = (0, 2), v_2 = (-2, 1)\}$.

$$\begin{aligned} f(1, 2) = (3, -1) &= \lambda_1 v_1 + \lambda_2 v_2 \\ &= \lambda_1(0, 2) + \lambda_2(-2, 1) \\ &= (0, 2\lambda_1) + (-2\lambda_2, \lambda_2) \\ &= (-2\lambda_2, 2\lambda_1 + \lambda_2) \end{aligned}$$

$-2\lambda_2 = 3 \Rightarrow \lambda_2 = \frac{-3}{2}$

So,

$2\lambda_1 + \lambda_2 = 2\lambda_1 + \frac{-3}{2} = -1 \Rightarrow \lambda_1 = \frac{1}{4}$.

Then $(\lambda_1, \lambda_2) = (\frac{1}{4}, \frac{-3}{2})$.

Now,

$f(-1, 1) = (0, -2) = \lambda_1 v_1 + \lambda_2 v_2$

$$= \lambda_1(0,2) + \lambda_2(-2,1)$$
$$= (0, 2\lambda_1) + (-2\lambda_2, \lambda_2)$$
$$= (-2\lambda_2, 2\lambda_1 + \lambda_2)$$

$-2\lambda_2 = 0 \Rightarrow \lambda_2 = 0$

So,

$2\lambda_1 + \lambda_2 = 2\lambda_1 = -2 \Rightarrow \lambda_1 = -1.$

Then $(\lambda_1, \lambda_2) = (-1, 0)$.

We find:

$$\mathcal{M}_{(B,B')}(g) = \begin{pmatrix} \frac{1}{4} & -1 \\ \frac{-3}{2} & 0 \end{pmatrix}$$

Proposition 3.5.4

Let E and F be two $\mathbb{K}$-vector spaces of dimensions n and m, and $B = (e_1, e_2, ..., e_n)$ a basis of E, and $B^{'} = (v_1, v_2, ..., v_m)$ a basis of F. Then, the matrix $A \in \mathcal{M}_{(n,m)}(\mathbb{K})$ corresponds to a unique linear map f from E to F. The matrix representation of f with respect to basis B and $B^{'}$ is given by A.

Example 3.5.5

Consider the matrix $A = \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}$. We have:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + 4y \\ 2x - 3y \end{pmatrix}.$$

The associated linear application is:

$$f : \mathbb{R}^2 \to \mathbb{R}^2,$$
$$(x, y) \mapsto f(x, y) = (-x + 4y, 2x - 3y).$$

Note 3.5.6

If $\mathbb{R}^n$ and $\mathbb{R}^m$ are equipped with their canonical bases, the linear map f from $\mathbb{R}^n$ to $\mathbb{R}^m$ associated with a matrix $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$ is given by:

$$f(x_1, x_2, ..., x_n) = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Example 3.5.7

Consider the matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix}$. We have:

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y \\ y - 2z \\ -x + z \end{pmatrix}.$$

The associated linear map is:

$$f : \mathbb{R}^3 \to \mathbb{R}^3,$$
$$(x, y, z) \mapsto f(x, y, z) = (2x - y, y - 2z, -x + z).$$

Theorem 3.5.8

Let E, F, and G be IK-vector spaces with basis B, $B^{'}$, and $B^{''}$, respectively.
If $f : E \to F$ and $g : F \to G$ are linear maps, then:

$$\mathcal{M}(B, B^{''})(g \circ f) = \mathcal{M}(B^{'}, B^{''})(g) \cdot \mathcal{M}(B, B^{'})(f).$$

Example 3.5.9

Consider linear applications:

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R}^2, \\ (x, y, z) &\mapsto f(x, y, z) = (2x + y - z, 2x - y). \end{aligned}$$

And

$$\begin{aligned} g : \mathbb{R}^2 &\to \mathbb{R}^2, \\ (x, y) &\mapsto g(x, y) = (x - 2y, x + y). \end{aligned}$$

The matrix representations are:

$$\mathcal{M}(B, B^{''})(g \circ f) = \begin{pmatrix} -2 & 3 & -1 \\ 4 & 0 & -1 \end{pmatrix}$$

$$\mathcal{M}(B^{'}, B^{''})(g) = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$$

$$\mathcal{M}(B, B^{'}) = \begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\mathcal{M}(B^{'}, B^{''})(g) \cdot \mathcal{M}(B, B^{'})(f) = \mathcal{M}(B, B^{''})(g \circ f)$$

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 3 & -1 \\ 4 & 0 & -1 \end{pmatrix}$$

Theorem 3.5.10

For linear application $f : E \to F$, where B is a basis of E and $B^{'}$ is a basis of F:

$$f \text{ is bijective} \Leftrightarrow \det\Big(\mathcal{M}_{(B,B^{'})}(f)\Big) \neq 0.$$

In this case:

$$\mathcal{M}_{(B,B^{'})}(f^{-1}) = (\mathcal{M}_{(B,B^{'})}(f))^{-1}.$$

Example 3.5.11

Consider the linear application:

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R}^2, \\ (x, y) &\mapsto f(x, y) = (-x + y, x + y). \end{aligned}$$

Show that f is bijective and calculate its inverse.
The matrix representation is:

$$\mathcal{M}_2(f) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The inverse is given by:

$$(\mathcal{M}_2(f^{-1}))^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Therefore, $f^{-1}(x,y) = \left(\frac{1}{2} - \frac{1}{2}y, -\frac{1}{2}x - \frac{1}{2}y\right)$.

3.6 Invariance of rank under transposition

Proposition 3.6.1

If $A \in \mathcal{M}_{(n,m)}(\mathbb{K})$ is associated with a linear map $f : E \to F$, where B is a basis of E and B' is a basis of F, then:

$$\text{rank}(A) = \text{rank}(f).$$

Also,

$$\text{rank}(A) = \text{rank}(A\,^t)$$

3.7 Matrices and Change of Basis

Definition 3.7.1

Let E be a vector space, and let $B = (e_1, e_2, \ldots, e_n)$ and $B' = (e'_1, e'_2, \ldots, e'_n)$ be two basis for E. The change of basis matrix from B' to B is, by definition, the matrix $\det\left(\mathcal{M}_{(B,B')}(Id_E)\right)$,
where Id_E is the identity map:

$$\begin{aligned} Id_E : E &\to E \\ x &\mapsto Id'_E(x) = x. \end{aligned}$$

The basis vectors of B can be expressed in B' according to the relations:

$$S : \begin{cases} e_1 = a_{11}e'_1 + a_{12}e'_2 + a_{13}e'_3 + \cdots + a_{1n}e'_n \\ e_2 = a_{21}e'_1 + a_{22}e'_2 + a_{23}e'_3 + \cdots + a_{2n}e'_n \\ e_3 = a_{31}e'_1 + a_{32}e'_2 + a_{33}e'_3 + \cdots + a_{3n}e'_n \\ \vdots \\ e_n = a_{n1}e'_1 + a_{n2}e'_2 + a_{n3}e'_3 + \cdots + a_{nn}e'_n. \end{cases}$$

The matrix of passage from B' to B is the square matrix P defined by:

$$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Example 3.7.2

$B' = (e'_1, e'_2, e'_3)$, $e'_1 = (1,1,1)$, $e'_2 = (1,1,0)$, $e'_3 = (1,0,0)$ and $B = (e_1, e_2, e_3)$ is the canonical basis of $\mathbb{R}^3$.

$$\begin{aligned} Id_E : \mathbb{R}^3_B &\to \mathbb{R}^3_{B'} \\ (x,y,z) &\mapsto Id_{\mathbb{R}^3}(x,y,z) = (x,y,z) \end{aligned}$$

$\mathcal{M}_{(B,B')}(Id_{\mathbb{R}^3}) = ?$

$$S : \begin{cases} Id\ (e_1) = (1,0,0) = a_{11}e'_1 + a_{12}e'_2 + a_{13}e'_3 \\ Id\ (e_2) = (0,1,0) = a_{21}e'_1 + a_{22}e'_2 + a_{23}e'_3 \\ Id\ (e_3) = (0,0,1) = a_{31}e'_1 + a_{32}e'_2 + a_{33}e'_3 \end{cases}$$

$$S \Leftrightarrow \begin{cases} Id\ (e_1) = (1,0,0) = a_{11}(1,1,1) + a_{12}(1,1,0) + a_{13}(1,0,0) \\ Id\ (e_2) = (0,1,0) = a_{21}(1,1,1) + a_{22}(1,1,0) + a_{23}(1,0,0) \\ Id\ (e_3) = (0,0,1) = a_{31}(1,1,1) + a_{32}(1,1,0) + a_{33}(1,0,0) \end{cases}$$

$$S \Leftrightarrow \begin{cases} a_{11} = \frac{1}{2}, & a_{12} = -\frac{1}{2}, & a_{13} = \frac{1}{2} \\ a_{21} = \frac{1}{2}, & a_{22} = \frac{1}{2}, & a_{23} = -\frac{1}{2} \\ a_{31} = -\frac{1}{2}, & a_{32} = \frac{1}{2}, & a_{33} = \frac{1}{2} \end{cases}$$

Therefore:

$$\mathcal{M}_{(B,B')}(Id_{\mathbb{R}^3}) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

Proposition 3.7.3

The change of basis matrix from a basis B to a basis B' is the inverse matrix of the change of basis matrix from B' to B:

$$\mathcal{M}_{(B',B)}(Id_{\mathbb{R}^3}) = \left(\mathcal{M}_{(B,B')}(Id_{\mathbb{R}^3})\right)^{-1}.$$

Note 3.7.4

Let E be a vector space, and let $B = (e_1, e_2, \ldots, e_n)$ and $B' = (e'_1, e'_2, \ldots, e'_n)$ be two basis for E. The basis vectors of B' can be expressed in B according to the relations:

$$S : \begin{cases} e'_1 = b_{11}e_1 + b_{12}e_2 + a_{13}e_3 + \cdots + b_{1n}e_n \\ e'_2 = b_{21}e_1 + b_{22}e_2 + b_{23}e_3 + \cdots + b_{2n}e_n \\ e'_3 = b_{31}e_1 + b_{32}e_2 + b_{33}e_3 + \cdots + b_{3n}e_n \\ \vdots \\ e'_n = b_{n1}e_1 + b_{n2}e_2 + b_{n3}e_3 + \cdots + b_{nn}e_n \end{cases}$$

The change of basis matrix from B to B' is the square matrix P^{-1} defined by:

$$P^{-1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix}$$

Example 3.7.5

$$\mathcal{M}_{(B,B')}(Id_{\mathbb{R}^3}) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\mathcal{M}_{(B',B)}(Id_{\mathbb{R}^3}) = \left[\frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}\right]^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Change of basis matrix from basis B to B'.

Theorem 3.7.6

Let $f : E \to F$, B_1, B_1' be two basis for E, B_2, B_2' bases of F.
If P denotes the change of basis matrix from B_1 to B_1', and Q denotes the change of basis matrix from B_2 to B_2', then

$$\mathcal{M}_{(B_1',B_2')}(f) = Q^{-1}\mathcal{M}_{(B_1,B_2)}(f)P.$$

Example 3.7.7

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$(x, y, z) \mapsto (x + y + z, x - y)$$

We equip $\mathbb{R}^3$ with the canonical basis $B_3 = (e_1, e_2, e_3)$ and $\mathbb{R}^2$ with the canonical basis $B_2 = (v_1, v_2)$. The matrix representation $\mathcal{M}(B_3, B_2)(f)$ is given by

$$\mathcal{M}(B_3, B_2)(f) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

We also consider another basis $B_3' = (e_1', e_2', e_3')$ for $\mathbb{R}^3$ with $e_1' = (1, 0, 1)$, $e_2' = (1, 1, 0)$, $e_3' = (0, 1, 1)$.
For $\mathbb{R}^2$, we have the basis $B_2' = (v_1', v_2')$ with $v_1' = (-1, 1)$ and $v_2' = (1, 1)$.
The change of basis matrices are given by

$$P = \mathcal{M}_{(B_3',B_3)}(\mathrm{Id}_{\mathbb{R}^3}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$Q = \mathcal{M}_{(B_2',B_2)}(\mathrm{Id}_{\mathbb{R}^2}) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

The linear transformation f can be represented as

$$\mathcal{M}_{(B_2',B_3')}(f) = Q^{-1}_{(B_2,B_2')}P = Q^{-1}M_{(B_3,B_2')}(f)P$$

Substituting the matrices, we get

$$\mathcal{M}_{(B_2',B_3')}(f) = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & -1 & -1/2 \\ 3/2 & 1 & 1/2 \end{bmatrix}$$

3.8 Diagonalization

Definition 3.8.1

Let $A \in M(n, n)(\mathbb{K})$ and $\lambda \in \mathbb{K}$. We say that λ is an eigenvalue of A if there exists a non-zero column vector v such that $Av = \lambda v$. The vector v is called the eigenvector associated with the eigenvalue λ.

Example 3.8.2

Consider the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$.
For $\lambda_1 = 1$, the corresponding eigenvector is

$$v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

and for $\lambda_2 = 2$, the corresponding eigenvector is

$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Proposition 3.8.3

Let $A \in M(n,n)(\mathbb{K})$ and $\lambda \in \mathbb{K}$. λ is an eigenvalue of A if and only if $P_A(\lambda) = \det(A - \lambda I_n) = 0$. Here, $P_A(\lambda)$ is called the characteristic polynomial of A.

Example 3.8.4

For the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

The characteristic polynomial is

$$P_A(\lambda) = \det(A - \lambda I_2) = (2-\lambda)(1-\lambda) = 0$$

Then

$$\begin{aligned} P_A(\lambda) &= \det\left(\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \det\left(\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) \\ &= \det\left(\begin{pmatrix} 2-\lambda & 2 \\ 0 & 1-\lambda \end{pmatrix}\right) \\ &= (2-\lambda)(1-\lambda) = 0 \end{aligned}$$

This gives the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$.

Definition 3.8.5

Let $A \in M(n,n)(\mathbb{K})$ and $\lambda \in \mathbb{K}$. If λ is an eigenvalue of A, then the set

$$E = \{v \in \mathbb{R}^n \text{ or } \mathbb{C}^n : Av = \lambda v\}$$

is called the eigenspace associated with the eigenvalue λ. The set E is a subspace of $\mathbb{R}^n$ or $\mathbb{C}^n$.

Example 3.8.6

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

The eigenvalues of A are 2 (double) and 1 (simple). For $\lambda = 2$, the eigenspace E_2 is spanned by the vectors $(1,1,0)$ and $(0,1,1)$. For $\lambda = 1$, the eigenspace E_1 is spanned by the vector $(1,1,0)$.

Definition 3.8.7

A matrix $A \in \mathcal{M}_n(IK)$ is said to be diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ (where P is the change of basis matrix).

Theorem 3.8.8

Let $A \in \mathcal{M}_n(IK)$, and let $\lambda_1, \lambda_2, ..., \lambda_n \in IK$ be the eigenvalues of A with respective multiplicities $m_1, m_2, ..., m_p$. Then, if either:

1. $dim E_{\lambda_i} = m_i$ for $i = 1, 2, ..., p$. or
2. $dim E_{\lambda_1} + dim E_{\lambda_2} + ... + dim E_{\lambda_p} = n$.

Then, matrix A is diagonalizable, and the associated diagonal matrix D is given by:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \lambda_p & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \lambda_n \end{bmatrix}$$

Each entry repeats m_i times, and matrix P is formed by the eigenvectors.

Note 3.8.9

If matrix $A \in \mathcal{M}_n(IK)$ has n distinct eigenvalues, then A is diagonalizable, and the associated diagonal matrix D is:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

Example 3.8.10

Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. It has eigenvalues $\lambda_1 = 2$ (double) and $\lambda_2 = 1$ (simple). The diagonal matrix D is:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

And the change of basis matrix P is:

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

3.9 Exercises

Exercise 3.9.1

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

1. Calculate A^2 and A^3. Evaluate $A^3 - A^2 + A - I$.
2. Express A^{-1} in terms of A^2, A, and I.
3. Express A^4 in terms of A^2, A, and I.

Exercise 3.9.2

Let $A = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{bmatrix}$

Calculate $(A - 2I)^3$, then deduce that A is invertible and find A^{-1} in terms of I, A, and A^2.

Exercise 3.9.3

Consider the matrix A defined as:

$$A = \begin{bmatrix} 5 & 6 & -3 \\ -18 & -19 & 9 \\ -30 & -30 & 14 \end{bmatrix}$$

1. Is A invertible? If yes, determine its inverse A^{-1}.
2. Calculate $A^2 - A - 2I_3 = 0$, where I_3 is the identity matrix.

Exercise 3.9.4

Consider the matrix A defined as:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

1. Find a, $b \in \mathbb{R}$ such that $A^2 = aI_3 + bA$.
2. Deduce that A is invertible and determine its inverse.

Exercise 3.9.5

Consider the matrix associated with the linear transformation f defined on $\mathbb{R}^3$ with respect to the canonical basis:

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

1. Determine the linear application f.
2. Find $ker\ f$ and $Im\ f$, along with their dimensions. Is f bijective?
3. Let $S = \{v_1 = (1, 1, 1), v_2 = (1, 0, 1), v_3 = (2, -1, 0)\}$:
   (a) Show that S is a basis for $\mathbb{R}^3$.
   (b) Find the matrix associated with f with respect to the basis S.

Exercise 3.9.6

Consider the matrix A defined as:

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$

1. Determine the eigenvalues of A.
2. Show that A is diagonalizable.
3. Find P and calculate A^k.

Exercise 3.9.7

Let be the following matrices

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -1 & 1 \\ 1 & 2 & -4 \\ -2 & 1 & 3 \end{pmatrix}$$

1. Calculate $A \times B$
2. Calculate A^t, B^t and $(AB)^t$
3. Calculate $Det\ (AB)$ and $Tr\ (AB)$

Exercise 3.9.8

Let

$$f\ (x, y) = (-x,\ 2x + y,\ 3x + y)$$

$$g\ (x, y, z) = (x\ - y,\ x + z,\ y\ - 2z\ )$$

1. Find the associated matrix of the linear application f according to the canonical basis for the starting set and the arrival set.
2. Find the associated matrix of the linear application g according to the basis $\beta = \{(1; 0; 0),\ (1; 1; 0),\ (1; 1; 1)\}$ for the starting set and the canonical basis for the arrival set.
3. Find the associated matrix of the linear application $g \circ f$

3.10 Solution of exercises

Solution 3.9.1

1. • $A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

• $A^3 = A^2 \times A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

• $A^3 - A^2 + A - I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$

2. $A^3 - A^2 + A - I = O \Leftrightarrow A(A^2 - A + I) = I \quad \text{donc } A^{-1} = A^2 - A + I$

3. $A^3 = A^2 - A + I$ donc $A^4 = A(A^2 - A + I) = A^3 - A^2 + A = (A^2 - A + I\ ) - A^2 + A = I$

Solution 3.9.2

$$(A - 2I) = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix}.$$

$$(A - 2I)^2 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

$$(A - 2I)^3 = (A - 2I)^2 \times (A - 2I) = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O.$$

This implies that:
$A^3 - 3 \times 2A^2 + 3 \times 2^2 A - 2^3 I = O$
Because A and I commute.
Which is equivalent to
$A^3 - 6A^2 + 12A - 8I = O$
Or
$A^3 - 6A^2 + 12A = 8I$
Then by dividing by 8 and factoring out A
$A(\frac{1}{8}A^2 - \frac{3}{4}A + \frac{3}{2}I) = I$
This shows that A is invertible and $A^{-1} = (\frac{1}{8}A^2 - \frac{3}{4}A + \frac{3}{2}I)$

Solution 3.9.3

1. A is invertible if and only if $detA \neq 0$.

$$|A| = \left|\begin{pmatrix} 5 & 6 & -3 \\ -18 & -19 & 9 \\ -30 & -30 & 14 \end{pmatrix}\right| \overset{C_2:C_2-C_1}{=} \left|\begin{pmatrix} 5 & 1 & -3 \\ -18 & 1 & 9 \\ -30 & 0 & 14 \end{pmatrix}\right| \overset{C_3:C_3+6C_2}{=} \left|\begin{pmatrix} 5 & 1 & -3 \\ -18 & 1 & 9 \\ 0 & 6 & 68 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} 5 & 1 & -3 \\ 0 & 25 & -45 \\ 0 & 6 & 68 \end{pmatrix}\right| = \left|\begin{pmatrix} 5 & 1 & -3 \\ 0 & 1 & -\frac{9}{5} \\ 0 & 6 & 68 \end{pmatrix}\right| = \left|\begin{pmatrix} 5 & 0 & -\frac{6}{5} \\ 0 & 1 & -\frac{9}{5} \\ 0 & 0 & \frac{242}{5} \end{pmatrix}\right| = 5 \times 1 \times \frac{242}{5} = 242 \neq 0.$$

Hence, A is invertible.

2. Let's calculate $A^2 - A - 2I_3 = 0$.

$$A^2 = A \cdot A = \begin{bmatrix} 7 & 6 & -3 \\ -18 & -17 & 9 \\ -30 & -30 & 16 \end{bmatrix}$$

$$A^2 - A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2I$$

So, $A^2 - A - 2I_3 = 0$.

We observe that $A(A - I_3) = 2I_3 \Rightarrow A\left(\frac{1}{2}A - \frac{1}{2}I_3\right) = I_3 \Rightarrow A^{-1} = \left(\frac{1}{2}A - \frac{1}{2}I_3\right)$.

Solution 3.9.4

1. Find a, $b \in \mathbb{R}$ such that $A^2 = aI_3 + bA$.

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}$$

Therefore, $a = 2$ and $b = 1$.

2. $\det(A) = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 2 \neq 0$, hence A is invertible.

$$A^2 = 2I_3 + A \Rightarrow A^2 - A = 2I_3 \Rightarrow A\left(\frac{1}{2}A - \frac{1}{2}I_3\right) = I_3 \Rightarrow A^{-1} = \frac{1}{2}A - \frac{1}{2}I_3.$$

Solution 3.9.5

1. Determine the function f.

$$f(x, y, z) = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x - y + 5z, 3x + 2z, x + y + 4z)$$

2. $\ker f = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = (0, 0, 0)\}$

So, $\dim \ker f = 0$ (since f is injective).

$\operatorname{Im} f = \{f(x, y, z) \mid (x, y, z) \in \mathbb{R}^3\}$

$$\operatorname{Im} f = \{x(1, 3, 1) + y(-1, 0, 1) + z(5, 2, 4) \mid x, y, z \in \mathbb{R}\}$$

The family $\{(1, 3, 1), (-1, 0, 1), (5, 2, 4)\}$ is linearly independent because $\det((1, 3, 1), (-1, 0, 1), (5, 2, 4)) = 23 \neq 0$. Therefore, $\dim \operatorname{Im} f = 3$, and f is surjective, implying that f is bijective.

3. Let $S = \{v_1 = (1, 1, 1), v_2 = (1, 0, 1), v_3 = (2, -1, 0)\}$.

a) S is a basis of $\mathbb{R}^3 \Leftrightarrow \det(v_1, v_2, v_3) \neq 0$

$$\det(v_1, v_2, v_3) = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} \overset{C_3 = C_1 + C_3}{=} \begin{vmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 2 \neq 0$$

b) Let A' be the matrix associated with f using the basis S. $A' = P^{-1}AP$, with

$$P = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Let's perform a change of basis using $P^{-1} = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{-1}{2} & -1 & \frac{3}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$.

$$A' = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{-1}{2} & -1 & \frac{3}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 5 \\ 3 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 5 & -10 \\ \frac{7}{2} & 2 & -\frac{9}{2} \\ 8 & 5 & -8 \end{bmatrix}$$

Solution 3.9.6

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

1. Determine the eigenvalues of A, denoted as $\lambda \in \mathbb{R}$

$$P_A(\lambda) = |A - \lambda I_3| = \begin{vmatrix} 0-\lambda & 2 & -1 \\ 3 & -2-\lambda & 0 \\ -2 & 2 & 1-\lambda \end{vmatrix} = 0$$

   The eigenvalues are 1, 2, and -4.

2. A is diagonalizable because it has three distinct eigenvalues.

3. Find the eigenvectors.

   • For $\lambda = 1$

$$E_1 = \{v = (x, y, z) \in \mathbb{R}^3 \mid Av = v\} \Rightarrow \begin{cases} 2y - z = x \\ 3x - 2y = y \\ -2x + 2y + z = z \end{cases} \Rightarrow \begin{cases} -x + 2y - z = 0 \\ 3x - 3y = 0 \\ -2x + 2y = 0 \end{cases} \Rightarrow x = y = z$$

   So, $E_1 = \{x(1, 1, 1) \mid x \in \mathbb{R}\}$, where $v_1 = (1, 1, 1)$ is the eigenvector associated with $\lambda = 1$.

   • For $\lambda = 2$

$$E_2 = \{v = (x, y, z) \in \mathbb{R}^3 \mid Av = 2v\} \Rightarrow \begin{cases} 2y - z = 2x \\ 3x - 2y = 2y \\ -2x + 2y + z = 2z \end{cases} \Rightarrow \begin{cases} -2x + 2y - z = 0 \\ 3x - 4y = 0 \\ -2x + 2y - z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{4}{3}y \\ z = \frac{-2}{3}y \end{cases}$$

   So, $E_2 = \{y(\frac{4}{3}, 1, \frac{-2}{3}) \mid y \in \mathbb{R}\}$, where $v_2 = (4, 3, -2)$ is the eigenvector associated with $\lambda = 2$.

   • For $\lambda = -4$

$$E_{-4} = \{v = (x, y, z) \in \mathbb{R}^3 \mid Av = -4v\} \Rightarrow \begin{cases} 2y - z = -4x \\ 3x - 2y = -4y \\ -2x + 2y + z = -4z \end{cases} \Rightarrow \begin{cases} 4x + 2y - z = 0 \\ 3x + 2y = 0 \\ -2x + 2y + 5z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{-2}{3}y \\ z = x \end{cases}$$

So, $E_{-4} = \{x(1, \frac{-3}{2}, 1) \mid x \in \mathbb{R}\}$, where $v_3 = (2, -3, 2)$ is the eigenvector associated with $\lambda = -4$.
Thus,

$$P = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 3 & -3 \\ 1 & -2 & 2 \end{bmatrix}$$

and $A = PDP^{-1}$, with

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

and

$$P^{-1} = \frac{1}{\det P} C_p = \frac{1}{-30} \begin{pmatrix} 0 & -12 & -18 \\ -5 & 0 & 5 \\ -5 & 6 & -1 \end{pmatrix} \Rightarrow C_p^t = \begin{pmatrix} 0 & -5 & -5 \\ -12 & 0 & 6 \\ -18 & 5 & -1 \end{pmatrix}$$

We compute P^{-1} by using the change of basis.
Then

$$A^k = PD^k P^{-1} = \begin{pmatrix} (-5)2^{k+2} - 10(-4)^k & -12 + 12(-4)^k & -18 + (5)2^{k+2} - 2(-4)^k \\ (-15)2^k - 15(-4)^k & -12 + 15(-4)^k & -18 + (5)2^{k+1} + 3(-4)^k \\ (5)2^{k+1} - 10(-4)^k & -12 + 15(-4)^k & -18 + (5)2^{k+1} - 2(-4)^k \end{pmatrix}$$

Solution 3.9.7

Let be the following matrices

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -1 & 1 \\ 1 & 2 & -4 \\ -2 & 1 & 3 \end{pmatrix}$$

Calculate $A \times B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 4 & -1 & 1 \\ 1 & 2 & -4 \\ -2 & 1 & 3 \end{pmatrix}$

First row

$c_{11} = 1(4) + 1(1) + 2(-2) = 1,$

$c_{12} = 1(-1) + 1(2) + 2(1) = 3,$

$c_{13} = 1(1) + 1(-4) + 2(3) = 3.$

Second row

$c_{21} = 0(4) + (-1)(1) + 1(-2) = -3,$

$c_{22} = 0(-1) + (-1)(2) + 1(1) = -1,$

$c_{23} = 0(1) + (-1)(-4) + 1(3) = 7.$

Third row

$c_{31} = 1(4) + 2(1) + 1(-2) = 4,$

$c_{32} = 1(-1) + 2(2) + 1(1) = 4,$

$c_{33} = 1(1) + 2(-4) + 1(3) = -4.$

Hence

$$AB = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -1 & 7 \\ 4 & 4 & -4 \end{pmatrix}$$

2. Calculate A^t, B^t, and $(AB)^t$

Transpose of A

$$A^t = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

Transpose of B

$$B^t = \begin{pmatrix} 4 & 1 & -2 \\ -1 & 2 & 1 \\ 1 & -4 & 3 \end{pmatrix}$$

Transpose of AB

$$(AB)^t = \begin{pmatrix} 1 & -3 & 4 \\ 3 & -1 & 4 \\ 3 & 7 & -4 \end{pmatrix}$$

(Indeed, $(AB)^t = B\,{}^tA^t$)

3. Calculate $\det(AB)$ and $Tr\ (AB)$

• Determinant

Using

$$\det(AB) = \det(A) \times \det(B),$$

First compute:

$$\det(A) = \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 0$$

The matrix A is not invertible.

therefore,

$$\det(AB) = \det(A) \times \det(B) = 0 \times \det(B) = 0$$

So,

$$\det(AB) = 0$$

• Trace

The trace is the sum of the diagonal entries of AB

$$Tr(AB) = 1 + (-1) + (-4) = -4$$

So,

$$Tr(AB) = -4$$

Solution 3.9.8

Given

$$f\ (x, y) = (-x,\ 2x + y,\ 3x + y)$$

$$g\ (x, y, z) = (x\ - y,\ x + z,\ y\ - 2z\ )$$

1. Matrix of f in canonical bases

We use the canonical basis:

$$e_1 = (1,0), \qquad e_2 = (0,1)$$

Compute:

Image of e_1

$$f\ (1,0) = (-1,2,3)$$

Image of e_2

$$f\ (0,1) = (0,1,3)$$

So the matrix of f is:

$$A_f = \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 3 & 3 \end{pmatrix}$$

2) Matrix of g in basis β Basis:

$$\beta = \{v_1 = (1,0,0), \quad v_2 = (1,1,0), \quad v_3 = (1,1,1)\}$$

2. We compute $g(v_i)$ and express in canonical basis (since arrival is canonical).

Step1: $g(v_1)$

$$g(1,0,0) = (1,1,0)$$

Step2: $g(v_2)$

$$g(1,1,0) = (0,1,1)$$

Step3: $g(v_3)$

$$g(1,1,1) = (0,2,-1)$$

3) Matrix of $g \circ f$

We use:

$$g \circ f = g(f)$$

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$

$$A_f = \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 3 & 3 \end{pmatrix}$$

$$A_g = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

Multiply $A_g A_f$

$$A_{g\circ f} = A_g \times A_f$$

$$A_{g\circ f} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 3 & 3 \end{pmatrix}$$

First column

$(1, -1, 0) = -1 - 2 + 0 = -3$

$(1, 1, 1) = -1 + 2 + 3 = 4$

$(0, 1, -2) = 2 - 6 = -4$

So first column is: $(-3, 4, -4)$

Second column

$(1, -1, 0) = 0 - 1 + 0 = -1$

$(1, 1, 1) = 0 + 1 + 1 = 2$

$(0, 1, -2) = 1 - 2 = -1$

So second column is: $(-1, 2, -1)$

$$A_{g\circ f} = \begin{pmatrix} -3 & -1 \\ 4 & 2 \\ -4 & -1 \end{pmatrix}$$

Chapter 4

Solution of Systems of Equations

4.1 Introduction

This chapter is devoted to the study of systems of linear equations, which represent one of the most important applications of linear algebra. Such systems naturally arise in many areas of mathematics, physics, engineering, and applied sciences, where they are used to model relationships between unknown variables under linear constraints.

The study begins with the general notion of a system of equations and its matrix representation, which allows a compact and structured formulation of the problem. This matrix approach makes it possible to analyze and solve systems using algebraic methods rather than purely computational procedures. In particular, the system AX=B plays a central role in this framework. A key concept introduced in this chapter is the rank of a system, which provides essential information about the existence and uniqueness of solutions. The rank allows us to classify systems according to their solvability and to understand their underlying structure. The chapter also presents Cramer's method, which offers an explicit formula for solving square systems when the determinant is non-zero. Different cases are discussed in order to handle both compatible and incompatible systems, depending on the dimensions and rank conditions. Finally, this chapter aims to equip the student with the fundamental techniques required to analyze and solve linear systems efficiently, while also reinforcing the connection between matrix theory and linear algebraic methods.

4.2 System of Equations

Definition 4.2.1

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
A system of n linear equations with p unknowns and coefficients in $\mathbb{K}$ is called:

$$S : \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1p}x_p = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2p}x_p = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3p}x_p = b_3 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{np}x_p = b_n \end{cases}$$

where $(x_j)_{j=1,..,p}$ are the unknowns, and (a_{ij}), $b_j \in IK$.

4.2.1 Matrix Representation

Definition 4.2.2

Let $A = (a_{ij})_{1\leq i\leq n,\ 1\leq j\leq p}$, $B = \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{pmatrix}$, and $X = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_p \end{pmatrix}$. The system (S) becomes

$$AX = B.$$

If f is a linear map from IK^p to IK^n with A being the matrix associated with f in the canonical bases, and if we denote $X = (x_1, ..., x_p)$ and $b = (b_1, ..., b_n)$, then the system (S) becomes $f(X) = B$.

4.2.2 Solution of the System $AX = B$

Definition 4.2.3

A solution of the system (S) is any element $X = (x_1, ..., x_p)$, satisfying the n equations of (S). This is equivalent to finding a vector X such that $AX = B$ or an element $X \in IK^p$ such that $f(X) = B$.

Example 4.2.4

$$\left\{ \begin{array}{l} x + 2y = 1 \\ 3x - y = 4 \\ x - y = -2 \end{array} \right. \Leftrightarrow \left(\begin{array}{cc} 1 & 2 \\ 3 & -1 \\ 1 & -1 \end{array} \right) \left(\begin{array}{c} x \\ y \end{array} \right) = (1, 4, -2)$$

4.3 Rank of a System of Equations

The rank of a linear system is the rank of the matrix $(a_{ij})_{1 \leq i \leq n, 1 \leq j \leq p}$. If r is the rank of the linear system (S), then $r \leq n$ and $r \leq p$.

4.4 Cramer's Method

Definition 4.4.1

The system (S) is said to be of Cramer if $n = p = r$, meaning that (S) is a system of n equations with n unknowns and $detA \neq 0$.

Theorem 4.4.2

Every Cramer system has a solution given by:

$$X = A^{-1}B.$$

Theorem 4.4.3

In a Cramer system, the solution is given by the formulas:

$$x_i = \frac{detA_i}{detA}, \qquad i = 1, ..., n.$$

Where A_i is the reduced matrix of A, obtained by replacing the i-th column with the vector B.

Example 4.4.4

1. Cramer's formulas for a system of two equations $\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \left(\begin{array}{c} x \\ y \end{array} \right) = \left(\begin{array}{c} e \\ f \end{array} \right)$ are as follows if the determinant satisfies $ad - bc \neq 0$:

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \left(\begin{array}{c} x \\ y \end{array} \right) = \left(\begin{array}{c} e \\ f \end{array} \right) \Leftrightarrow \left\{ \begin{array}{l} ax + by = e \\ cx + dy = f \end{array} \right. \Rightarrow x = \frac{\left| \begin{array}{cc} e & b \\ f & d \end{array} \right|}{\left| \begin{array}{cc} a & b \\ c & d \end{array} \right|}, \quad \text{and } y = \frac{\left| \begin{array}{cc} a & e \\ c & f \end{array} \right|}{\left| \begin{array}{cc} a & b \\ c & d \end{array} \right|}$$

For this system, it gives

$$\begin{cases} 2x+y=1 \\ 3x+7y=-2 \end{cases}$$

$$x=\frac{\begin{vmatrix} 1 & 1 \\ -2 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix}}=\frac{9}{11}\ , \qquad y=\frac{\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix}}=\frac{-7}{11}$$

2. Cramer's formulas for a system of three equations $AX=B \Leftrightarrow \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} j \\ k \\ l \end{pmatrix}$ are as follows if the determinant satisfies $\det A \neq 0$:

$$x=\frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad y=\frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad \text{and } z=\frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

For this system, it gives

$$\begin{cases} 2x-y+z=1 \\ x+y-z=2 \\ 3x+y+z=3 \end{cases} \Leftrightarrow \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$x=\frac{\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix}}=\frac{6}{6}=1 \qquad ,y=\frac{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix}}\frac{3}{6}=\frac{1}{2}, \quad \text{and} \quad z=\frac{\begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 3 & 1 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix}}\frac{-3}{6}=\frac{-1}{2}$$

4.4.1 Case when n = p and r < n:

Consider a system of n equations with n unknowns, but $rgA<n$, i.e., $detA=0$. In this case, extract a matrix M from A, knowing that it is the largest invertible square matrix, i.e., $detM\neq 0$, contained in A and of order r. This is called a submatrix, and the unknowns associated with M become principal unknowns, while the remaining $(n-r)$ unknowns become parameters or arbitrary values. Consider the following system:

$$\begin{cases} a_{11}x_1+a_{12}x_2+\ldots+a_{1r}x_r=b_1-(a_{1(r+1)}x_{r+1}+\ldots+a_{1n}x_n)=b'_1 \\ a_{21}x_1+a_{22}x_2+\ldots+a_{2r}x_r=b_2-(a_{1(r+1)}x_{r+1}+\ldots+a_{1n}x_n)=b'_2 \\ \ldots \\ a_{r1}x_1+a_{r2}x_2+\ldots+a_{rr}x_r=b_r-(a_{1(r+1)}x_{r+1}+\ldots+a_{1n}x_n)=b'_r \end{cases}$$

This last system is a Cramer system, so it has a unique solution $(x_1,...,x_r)$ depending on $(x_{r+1},...,x_n)$. If this solution satisfies the remaining $(n-r)$ equations, then the overall system has infinitely many solutions. If, on the other hand, $(x_1,...,x_r)$ does not satisfy any of the remaining $(n-r)$ equations, then the overall system has no solution.

Example 4.4.5

$$S_1:\begin{cases} 3x-y+2z=3 \\ 2x+2y+z=2 \\ x-3y+z=1 \end{cases} \Leftrightarrow \begin{pmatrix} 3 & -1 & 2 \\ 2 & 2 & 1 \\ 1 & -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$detA=det\begin{pmatrix} 3 & -1 & 2 \\ 2 & 2 & 1 \\ 1 & -3 & 1 \end{pmatrix}=0 \Rightarrow (S_1)$ is not a Cramer system since $det\begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}\neq 0 \Rightarrow rg(A)=2$, and we consider x, y as unknowns and z as a parameter. Thus, we obtain the system:

$$\begin{pmatrix}3 & -1\\ 2 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3-2z\\ 2-z\end{pmatrix}\Leftrightarrow\begin{cases}3x-y=3-2z\\ 2x+2y=2-z\end{cases}$$

which is a Cramer system and has a unique solution (x, y) depending on z.

$x=\frac{det\begin{pmatrix}3-2z & -1\\ 2-z & 2\end{pmatrix}}{det\begin{pmatrix}3 & -1\\ 2 & 2\end{pmatrix}}=\frac{8-5z}{8}$, $y=\frac{det\begin{pmatrix}3 & 3-2z\\ 2 & 2-z\end{pmatrix}}{det\begin{pmatrix}3 & -1\\ 2 & 2\end{pmatrix}}=\frac{z}{8}$

Now, check if (x, y) satisfies $x-3y+z=1$ (remaining equation): $\frac{8-5z}{8}-3(\frac{z}{8})+z=\frac{6}{8}\neq -5$. Therefore, the system has infinitely many solutions given by:

$\frac{(\frac{8-5z}{8},\frac{z}{8},z)}{z}\in\mathbb{R}$.

4.4.2 Case when $n \neq p$:

If the number of equations is not equal to the number of unknowns, determine the rank of A first and proceed as before. If M is a matrix contained in A and of order r with $detM \neq 0$, then consider the system of r equations with r unknowns corresponding to M, which is a Cramer system.

If the solution satisfies the remaining equations, then the overall system has infinitely many solutions. Otherwise, it has no solution.

Example 4.4.6

$$\begin{cases}3x-y=4\\ 2x+2y=3\\ x-5y=-5\end{cases}\Leftrightarrow\begin{pmatrix}3 & -1\\ 2 & 2\\ 1 & -5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}4\\ 3\\ -5\end{pmatrix}$$

The rank of $\begin{pmatrix}3 & -1\\ 2 & 2\\ 1 & -5\end{pmatrix}\leq 2$, choose $M=\begin{pmatrix}3 & -1\\ 2 & 2\end{pmatrix}\Rightarrow detM=8\neq 0\Rightarrow rg(M)=2$. Take the system:

$$\begin{cases}3x-y=4\\ 2x+2y=3\end{cases}\Rightarrow\begin{cases}x=\frac{11}{8}\\ y=\frac{1}{8}\end{cases}$$

Now, check the remaining equation:
$x-5y=-5\Rightarrow\frac{11}{8}-5(\frac{1}{8})=\frac{6}{8}\neq -5$. Therefore, the system has no solution.

4.5 Gaussian Elimination Method

Definition 4.5.1

Gaussian elimination is a systematic method for solving systems of linear equations. It consists in transforming the augmented matrix of the system into an equivalent simpler form using elementary row operations.

The goal is to obtain an upper triangular (row echelon) form, from which the solutions can be determined by back substitution.
A system is said to be equivalent if it has the same solution set after applying elementary row operations

4.5.1 Elementary Row Operations

The Gaussian elimination method is based on three types of operations:

Row swapping:
$R_i \leftrightarrow R_j$
Row multiplication:
$R_i \leftrightarrow \lambda R_i$, , where $\lambda \neq 0$
Row replacement:
$R_i \leftrightarrow R_i + \lambda R_j$
These operations do not change the solution set of the system.

4.5.2 Row Echelon Form

A matrix is in row echelon form if
- All zero rows (if any) are at the bottom
- The first non-zero entry of each row (called pivot) is to the right of the pivot in the previous row
- All entries below each pivot are zero.

Once this form is obtained, the system can be solved easily by back substitution

4.5.3 Algorithm of Gaussian Elimination

Given a system

$$AX = B$$

1) Write the augmented matrix $(A \mid B)$
2) Use elementary row operations to obtain an upper triangular matrix.
3) Solve the resulting system starting from the last equation
4) Express the solution in parametric form if necessary

Example 4.5.2

Solve the system:

$$\begin{cases} x+y+z=3 \\ 2x+3y+z=7 \\ x+2y+2z=6 \end{cases}$$

Step 1: Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 7 \\ 1 & 2 & 2 & 6 \end{array}\right)$$

Step 2: Elimination
- $R_2 \leftrightarrow R_2 - 2R_1$
- $R_3 \leftarrow R_3 - R_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right)$$

$R_3 \leftrightarrow R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

Step 3: Back substitution
From last row:
$2z = 2 \Rightarrow z = 1$
Second row
$y - z = 1 \Rightarrow y = 1 + z = 1 + 1 = 2$
First row
$x + y + z = 3 \Rightarrow x = 3 - y - z = 3 - 2 - 1 = 0$
Solution:

$$(x, y, z) = (0, 2, 1)$$

4.6 Rank Method for Solving Linear Systems

Definition 4.6.1

The rank method is a theoretical approach for studying and solving systems of linear equations using the concept of the rank of matrices. It is based on comparing the rank of the coefficient matrix with the rank of the augmented matrix.

Consider a linear system:

$$AX = B$$

where A is the coefficient matrix, X is the vector of unknowns, and B is the constant vector.

4.6.1 Definition of Rank

Definition 4.6.2

The rank of a matrix is the number of linearly independent rows (or columns). It represents the maximum number of independent equations in the system.

We denote:
$r\ (A)$: rank of coefficient matrix
$r\ (A\backslash B)$: rank of augmented matrix.

4.6.2 Rouché–Capelli Theorem

Theorem 4.6.3

A system

$$AX = B$$

has solutions if and only if:

$$r\ (A) = r\ (A\backslash B)$$

Moreover,
- If $r\ (A) = r\ (A\backslash B) = n$ (number of unknowns), the system has a unique solution.
- If $r(A) = r(A\backslash B) < n$, the system has infinitely many solutions.
- If $r(A) \neq r(A\backslash B)$, the system has no solution.

4.6.3 Method (Rank Method Procedure)

To solve a system using the rank method:
- Write the augmented matrix $(A\backslash B)$.
- Use elementary row operations to reduce it.
- Compute $r(A)$ and $r(A\backslash B)$.
- Apply Rouché–Capelli theorem.
- Deduce the type of solutions.
- If possible, express the solution parametrically.

Example 4.6.4

Solve:

$$\begin{cases} x + y + z = 3 \\ 2x + 2y + 2z = 6 \\ x - y + z = 1 \end{cases}$$

Step 1: Augmented matrix

$$\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 6 \\ 1 & -1 & 1 & 1 \end{array}$$

Step 2: Row reduction
- $R_2 \leftrightarrow R_2 - 2R_1$
- $R_3 \leftrightarrow R_3 - R_1$

$$\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 \end{array}$$

Step 3: Rank
$r\ (A) = 2$
$r\ (A \backslash B) = 2$
Since $n = 3$:
$r(A) = r(A \backslash B) = 2 < 3 \Rightarrow$ Infinite solutions
Step 4: Solve
From system:
$-2y = -2 \Rightarrow y = 1$
First equation:
$x + 1 + z = 3 \Rightarrow x = 2 - z$
Let $z = t$
So, the solution is:

$$(x, y, z) = (2 - t, 1, t), \quad t \in \mathbb{R}$$

4.7 Matrix Inverse Method

In this section, we use the matrix Inverse Method for Solving Linear Systems.

Definition 4.7.1

The matrix inverse method is an algebraic technique used to solve square systems of linear equations of the form:

$$AX = B$$

where: A is a square matrix of size $n \times n$,
X is the vector of unknowns,
B is the constant vector.
If the matrix A is invertible (i.e., $\det(A) \neq 0$), then the system has a unique solution given by:

$$X = A^{-1}B$$

4.7.1 Invertible Matrix

Definition 4.7.2

A square matrix
A is said to be invertible if there exists a matrix A^{-1} such that:

$$AA^{-1} = A^{-1}A = I$$

where
I is the identity matrix.
A matrix is invertible if and only if: $\det(A) \neq 0$

Important Theorem

Theorem 4.7.3

A linear system

$$AX = B$$

- has a unique solution if A is invertible
- no or infinitely many solutions if A is not invertible

In the invertible case:

$$X = A^{-1}B$$

4.7.2 Method (Steps)

To solve a system using the inverse matrix method:
- Write the system in matrix form $AX = B$.
- Compute $\det(A)$.
- If $\det(A) \neq 0$.
- compute A^{-1}.
- Multiply $A^{-1}B$.
- Obtain the solution vector X.

Example 4.7.4

Solve the system:

$$\begin{cases} x + y = 3 \\ 2x + 3y = 8 \end{cases}$$

Step 1: Matrix form

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$$

Step 2: Determinant
$\det(A) = 1 \cdot 3 - 2 \cdot 1 = 1 \neq 0$
So A is invertible
Step 3: Inverse of A
$A^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$
Step 4: Compute $X = A^{-1}B$
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
Final solution

$$(x, y) = (1, 2)$$

4.8 Exercises

Exercise 4.8.1

Solve the following system using Cramer's Method

$$S : \begin{cases} x + y + z = 3 \\ 2x + y + z = 2 \\ x + 2y + z = 1 \end{cases}$$

Exercise 4.8.2

Solve the following system using Cramer's Method

$$S : \begin{cases} 3x + y - 2z + 3t = 0 \\ -x + 2y - 4z + 6t = 2 \\ 2x - y + 2z - 3t = 0 \end{cases}$$

Exercise 4.8.3

Depending on the values of a, solve the following systems using Cramer's Method

$$S_1 : \begin{cases} ax + y = 2 \\ (a^2 + 1)x + 2ay = 1 \end{cases} \quad , \quad S_2 : \begin{cases} (a+1)x + (a-1)y = 1 \\ (a-1)x + (a+1)y = 1 \end{cases}$$

Exercise 4.8.4

Consider the system (S),

$$S : \begin{cases} x - my + m^2 z = 2m \\ mx - m^2 y + mz = 2m \\ mx + y - m^3 z = 1 - m \end{cases}$$

Solve (S), specifying the values of m for which it is a Cramer system.

Exercise 4.8.5

Solve the following system using Gaussian elimination

$$\begin{cases} x + y + z = 6 \\ 2x + 3y + z = 11 \\ x - y + 2z = 5 \end{cases}$$

Exercise 4.8.6

Solve the following system using Gaussian elimination

$$\begin{cases} x + 2y - z = 1 \\ 2x + 5y + z = 9 \\ 3x + 6y + 2z = 12 \end{cases}$$

Exercise 4.8.7

Solve using rank method

$$\begin{cases} x+y+z=3 \\ 2x+2y+2z=5 \\ x-y+z=0 \end{cases}$$

Exercise 4.8.8

Solve using rank method

$$\begin{cases} x+2y+z=4 \\ 2x+4y+2z=8 \\ x+y+z=3 \end{cases}$$

Exercise 4.8.9

Solve using the inverse matrix method

$$\begin{cases} 2x+y=5 \\ x+3y=7 \end{cases}$$

4.9 Solution of exercises

Solution 4.8.1

$$S:\begin{cases} x+y+z=3 \\ 2x+y+z=2 \\ x+2y+z=1 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$det\ A = det\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = 1 \neq 0,\ \ rg\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = n = p = 3.$$

Therefore, S is a Cramer system.

$$x = \frac{detA_1}{detA} = \frac{\begin{vmatrix} 3 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}} = -1 \ \ ,y = \frac{detA_2}{detA} = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}} = -2, \text{ and } \ z = \frac{detA_3}{detA} = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}} = 6$$

Solution 4.8.2

$$S:\begin{cases} 3x+y-2z+3t=0 \\ -x+2y-4z+6t=2 \\ 2x-y+2z-3t=0 \end{cases} \Leftrightarrow \begin{pmatrix} 3 & 1 & -2 & 3 \\ -1 & 2 & -4 & 6 \\ 2 & -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}=\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

The $rgA = rg\begin{pmatrix} 3 & 1 & -2 & 3 \\ -1 & 2 & -4 & 6 \\ 2 & -1 & 2 & 3 \end{pmatrix} = 3$. Take $M = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \Rightarrow \det M = 7 \neq 0$

Consider the following system:

$$\begin{cases} 3x+y=2z-3t \\ -x+2y=2+4z-6t \end{cases} \Rightarrow \begin{cases} x=\frac{-2}{7} \\ y=2z\ -3t+\frac{6}{7} \end{cases}$$

Substitute into the third equation:

$2x-y+2z-3t = \frac{-4}{7}\ -2z+3t\ -\frac{6}{7}+2z\ -3t = \frac{-10}{7} \neq 0$

Therefore, the system has no solution.

Solution 4.8.3

For system (S_1)

$$S_1:\begin{cases} ax+y=2 \\ (a^2+1)x+2ay=1 \end{cases} \Leftrightarrow \begin{pmatrix} a & 1 \\ a^2+1 & 2a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$det\ A = det\begin{pmatrix} a & 1 \\ a^2+1 & 2a \end{pmatrix} = a^2-1 \neq 0 \Rightarrow a \neq \pm 1.$$

So, there is a unique solution if and only if $a \neq \pm 1$.

Then:

$$x = \frac{detA_1}{detA} = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 2a \end{vmatrix}}{\begin{vmatrix} a & 1 \\ a^2+1 & 2a \end{vmatrix}} = \frac{4a-1}{a^2-1}\ ,\ \ y = \frac{detA_2}{detA} = \frac{\begin{vmatrix} a & 2 \\ a^2+1 & 1 \end{vmatrix}}{\begin{vmatrix} a & 1 \\ a^2+1 & 2a \end{vmatrix}} = \frac{-2a^2+a-2}{a^2-1}$$

• If $a=1$ then the system becomes: $\begin{cases} x+y=2 \\ 2x+2y=1 \end{cases} \Leftrightarrow \begin{cases} x+y=2 \\ x+y=\frac{1}{2} \end{cases}$

However, we cannot have both $x+y=2$ and $x+y=\frac{1}{2}$ at the same time. Therefore, there is no solution.

• If $a=-1$ then the system becomes: $\begin{cases} -x+y=2 \\ 2x-2y=1 \end{cases} \Leftrightarrow \begin{cases} -(x-y)=2 \\ x-y=\frac{1}{2} \end{cases}$ impossible, so there is no solution.

For system (S_2)

$$S_2:\begin{cases} (a+1)x+(a-1)y=1 \\ (a-1)x+(a+1)y=1 \end{cases} \Leftrightarrow \begin{pmatrix} a+1 & a-1 \\ a-1 & a+1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Here, the determinant is $\begin{vmatrix} a+1 & a-1 \\ a-1 & a+1 \end{vmatrix} = (a+1)^2-(a-1)^2 = 4a$

If $a \neq 0$ then we find a unique solution (x, y) with Cramer's formula

$$x = \frac{detA_1}{detA} = \frac{\begin{vmatrix} 1 & a-1 \\ 1 & a+1 \end{vmatrix}}{\begin{vmatrix} a+1 & a-1 \\ a-1 & a+1 \end{vmatrix}} = \frac{1}{2a}, \quad y = \frac{detA_2}{detA} = \frac{\begin{vmatrix} a+1 & 1 \\ a-1 & 1 \end{vmatrix}}{\begin{vmatrix} a+1 & a-1 \\ a-1 & a+1 \end{vmatrix}} = \frac{1}{2a}.$$

• If $a = 0$ then there is no solution.

Solution 4.8.4

$$S : \begin{cases} x - my + m^2 z = 2m \\ mx - m^2 y + mz = 2m \\ mx + y - m^3 z = 1 - m \end{cases} \Leftrightarrow \begin{pmatrix} 1 & -m & m^2 \\ m & -m^2 & m \\ m & 1 & -m^3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2m \\ 2m \\ 1-m \end{pmatrix}.$$

$$det\ A = det \begin{pmatrix} 1 & -m & m^2 \\ m & -m^2 & m \\ m & 1 & -m^3 \end{pmatrix} = m^5 - m = m(m^4 - 1) \neq 0 \Rightarrow m \neq 0 \text{ and } m \neq \pm 1.$$

• If $m \neq 0, 1, -1$, then (S) is a Cramer system and

$x = \frac{m(2m^2-3m+3)}{1+m^2}$, $y = \frac{(m-1)^2(2m+1)}{(1+m^2)(1+m)}$ and $z = \frac{2}{1+m}$.

Solution 4.8.5

Step 1: Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 1 & 11 \\ 1 & -1 & 2 & 5 \end{array}\right)$$

Step 2: Eliminate first column

• $R_2 \leftrightarrow R_2 - 2R_1$

$R_2 = (2, 3, 1 \mid 11) - 2(1, 1, 1 \mid 6) = (0, 1, -1 \mid -1)$

• $R_3 \leftrightarrow R_3 - R_1$

$R_3 = (1, -1, 2 \mid 5) - (1, 1, 1 \mid 6) = (0, -2, 1 \mid -1)$

New matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & 1 & -1 \end{array}\right)$$

Step 3: Eliminate second column

$R_3 \leftrightarrow R_3 + 2R_2$

$R_3 = (0, -2, 1 \mid -1) + 2(0, 1, -1 \mid -1) = (0, 0, -1 \mid -3)$

Matrix becomes:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & -3 \end{array}\right)$$

Step 4: Back substitution

From last row:

$-z = -3 \Rightarrow z = 3$

Second row:

$y - z = -1 \Rightarrow y - 3 = -1 \Rightarrow y = 2$

First row:

$x + y + z = 6 \Rightarrow x + 2 + 3 = 6 \Rightarrow x = 1$

Final Answer

$$(x, y, z) = (1, 2, 3)$$

Solution 4.8.6

Step 1: Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & 1 & 9 \\ 3 & 6 & 2 & 12 \end{array}\right)$$

Step 2: Eliminate first column
• $R_2 \leftarrow R_2 - 2R_1$
$R_2 = (2,5,1 \mid 9) - 2(1,2,-1 \mid 1) = (0,1,3 \mid 7)$
• $R_3 \leftarrow R_3 - 3R_1$
$R_3 = (3,6,2 \mid 12) - 3(1,2,-1 \mid 1) = (0,0,5 \mid 9)$
New Matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 5 & 9 \end{array}\right)$$

Step 3: Back substitution
From last row
$5z = 9 \Rightarrow z = \frac{5}{9}$
Second row:
$y + 3z = 7 \Rightarrow y + \frac{27}{5} = 7 \Rightarrow y = \frac{8}{5}$
Final Answer

$$(x,y,z) = (-\frac{2}{5}, \frac{8}{5}, \frac{9}{5}) = \frac{1}{5}(-2,8,9)$$

Solution 4.8.7

Solve the system using the rank method

$$\begin{cases} x + y + z = 3 \\ 2x + 2y + 2z = 5 \\ x - y + z = 0 \end{cases}$$

Step 1: Matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 5 \\ 1 & -1 & 1 & 0 \end{array}\right)$$

Step 2: Reduction
• $R_2 \leftrightarrow R_2 - 2R_1$
• $R_3 \leftrightarrow R_3 - R_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & -2 & 0 & -1 \end{array}\right)$$

Step 3: Rank
$r(A) = 2$
$r(A\backslash B) = 3$
Since: $r(A) = 2 \neq r(A\backslash B) \Rightarrow$ No solution.

Solution 4.8.8

Solve the system using the rank method

$$\begin{cases} x + 2y + z = 4 \\ 2x + 4y + 2z = 8 \\ x + y + z = 3 \end{cases}$$

solution

exo 2

Step 1: Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 4 & 2 & 8 \\ 1 & 1 & 1 & 3 \end{array}\right)$$

Step 2: Row reduction

• $R_2 \leftrightarrow R_2 - 2R_1$

$R_2 = (2, 4, 2 \mid 8) - 2(1, 2, 1 \mid 4) = (0, 0, 0 \mid 0)$

• $R_3 \leftrightarrow R_3 - R_1$

$R_3 = (1, 1, 1 \mid 3) - (1, 2, 1 \mid 4) = (0, -1, 0 \mid -1)$

New matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 \end{array}\right)$$

Step 3: Rank determination

Coefficient matrix A :

$$A = \left(\begin{array}{ccc} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{array}\right)$$

After reduction:

$$\left(\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right)$$

Two non-zero rows so,

$$r(A) = 2$$

Augmented matrix $(A \backslash B)$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 \end{array}\right)$$

Also two non-zero rows

$$r\ (A \backslash B) = 2$$

Step 4: Apply Rouché–Capelli theorem

$$r(A) = r(A \backslash B) = 2, \quad n = 3$$

So,

• system is compatible

• number of unknowns = 3

• rank = 2

Therefore,

infinitely many solutions

Step 5: Solve the system

From last row:

$-y = -1 \Rightarrow y = 1$

From first equation:

$x + 2(1) + z = 4 \Rightarrow x + z = 2 \Rightarrow x = 2 - z$

Let: $z = t$

Final solution

$$(x, y, z) = (2 - t, 1, t), \quad t \in \mathbb{R}$$

Solution 4.8.9

Solve the system using the inverse matrix method

$$\begin{cases} 2x + y = 5 \\ x + 3y = 7 \end{cases}$$

Step 1: Matrix form

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

Step 2: Determinant

$$\det(A) = 2 \cdot 3 - 1 \cdot 1 = 6 - 1 = 5 \neq 0$$

So A is invertible

Step 3: Inverse of A

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{-1}{5} \\ \frac{-1}{5} & \frac{2}{5} \end{pmatrix}$$

Step 4: Compute $X = A^{-1}B$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{-1}{5} \\ \frac{-1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{8}{5} \\ \frac{9}{5} \end{pmatrix}$$

Final answer

$$(x, y) = (\frac{8}{5}, \frac{9}{5})$$

Chapter 5

Examination Examples with Solutions

5.1 Introduction

This chapter is devoted to a collection of examination models designed to familiarize students with the structure and level of difficulty of typical written assessments in linear algebra and related mathematical tools. Unlike the previous chapters, which focus on theoretical foundations and methodological developments, this chapter emphasizes practice-oriented learning through exam-style problems. Each model presented in this chapter is carefully selected to reflect the types of questions commonly proposed in university examinations. The exercises cover the main topics studied throughout the course, allowing students to consolidate their knowledge and apply theoretical concepts in a structured and effective manner. A particular feature of this chapter is that each exercise is accompanied by a complete and detailed solution. These solutions are written in a clear and rigorous style, highlighting the logical steps required to reach the final answer. This approach is intended to guide students in developing proper mathematical reasoning and in improving the quality of their written work during examinations. In addition, a marking scheme is provided for each exercise. This allows students to understand how points are distributed across different parts of a question and to identify the key steps required to obtain full marks. This aspect is particularly useful for exam preparation, as it helps students become familiar with evaluation criteria and expectations. Overall, this chapter is designed as both a training tool and a preparation guide, aiming to strengthen the student's problem-solving skills, improve their exam performance, and build confidence in handling structured mathematical assessments.

5.2 Six Solved Sample Exams

Model Exam 1

? Exercise 1 (7 points)

Let:

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}.$$

1. Show that E is a vector subspace of $\mathbb{R}^3$
2. Find a basis of E.
3. Determine the dimension of E.

? Exercise 2 (7 points)

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by:

$$f(x, y, z) = (x + y,\ y + z,\ x + z).$$

1. Show that f is a linear application.
2. Determine the matrix of f in the canonical basis of $\mathbb{R}^3$
3. Find $ker(f)$ and $Im(f)$.
4. Is f isomorphism? Justify your answer.

? Exercise 3 (6 points)

Let A and B be two square matrices of the same order such that $AB = BA$

1. Prove that if A is invertible, then $A^{-1}BA = B$
2. If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

   determine the general form of A^n, then prove it by mathematical induction.

The Model Solution for the 1 Exam

Solution of Exercise 1

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}.$$

1. .....(3pts)

   • Contains $(0, 0, 0)$: yes

   Because $0 + 0 + 0 = 0$, so, $(0, 0, 0) \in E$

   • Closed under addition and scalar multiplication $\Rightarrow E$ is a subspace of $\mathbb{R}^3$.

2. .....(3pts)

   Solve:

   $x + y + z = 0 \Rightarrow z = -x - y$

   So:

   $(x, y, z) = (x, y, -x - y) = x(1, 0, -1) + y(0, 1, -1)$

   So the family $\{(1, 0, -1),\ (0, 1, -1)\}$ generates E

   Show that the family $\{(1, 0, -1),\ (0, 1, -1)\}$ is linearly independent

   Assume: $\forall\ a,\ b \in \mathbb{R}$

   $a(1, 0, -1) + b(0, 1, -1) = (0, 0, 0).$

   This gives:

   $(a,\ b,\ -a - b) = (0, 0, 0).$

   So:

   $a = 0,\ b = 0.$

   Thus the family is linearly independent.

   Then the basis of E is:

   $$\{(1, 0, -1),\ (0, 1, -1)\}$$

3. Dimension: .....(1pt)

   $$\dim(E) = 2$$

Solution of Exercise 2

$f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by:

$$f(x, y, z) = (x + y,\ y + z,\ x + z).$$

1. Linearity: let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ and $\lambda \in \mathbb{R}$ verify:.....(2pts)

   • Additivity

   $f(u + v) \stackrel{?}{=} f(u) + f(v)$

   $$\begin{aligned} f(u + v) &= f((x_1, y_1, z_1) + (x_2, y_2, z_2)) \\ &= f\ ((x_1 + x_2, y_1 + y_2, z_1 + z_2)) \\ &= (x_1 + x_2 + y_1 + y_2,\ y_1 + y_2 + z_1 + z_2,\ x_1 + x_2 + z_1 + z_2) \\ &= (x_1 + y_1,\ y_1 + z_1,\ x_1 + z_1) + (x_2 + y_2,\ y_2 + z_2,\ x_2 + z_2) \\ &= f(u) + f(v) \end{aligned}$$

   • Homogeneity

$f(\lambda u) \stackrel{?}{=} \lambda f(u)$

$$\begin{aligned} f(\lambda u) = f(\lambda(x_1, y_1, z_1)) &= f(\lambda x_1, \lambda y_1, \lambda z_1) \\ &= (\lambda x_1 + \lambda y_1, \lambda y_1 + \lambda z_1, \lambda x_1 + \lambda z_1) \\ &= \lambda(x_1 + y_1,\ y_1 + z_1,\ x_1 + z_1) \\ &= \lambda f(u) \end{aligned}$$

Therefore, f is a linear application.

2. Matrix: .....(2pts)

The canonical basis of $\mathbb{R}^3$ is

$e_1 = (1,0,0), \qquad e_2 = (0,1,0), \qquad e_3 = (0,0,1).$

Compute the images:

$f(e_1) = f(1,0,0) = (1,0,1)$

$f(e_2) = f(0,1,0) = (1,1,0)$

$f(e_3) = f(0,0,1) = (0,1,1)$

Thus, the matrix of f is

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

3. $Ker(f)$ and $Im(f)$.

• $Ker(f\ )$ .....(1pts)

We have $Ker\ f = \{(x,y,z) :\ f(x,y,z) = 0_{\mathbb{R}^3}\}$

We solve $f(x,y,z) = (0,0,0)$

$$(x+y,\ y+z,\ x+z) = (0,0,0)$$

$$\begin{cases} x + y = 0 \\ y + z = 0 \\ x + z = 0 \end{cases}$$

From the first equation we have: $y = -x$

From the second: $z = -y = x$

Substitute into the third: $2x = 0 \Rightarrow x = 0$

So, $x = 0 \Rightarrow y = 0 \Rightarrow z = 0$

Therefore, $Ker\ f = \{(0,0,0)\}$.

• $Im(f)$ .....(1pts)

Compute the determinant of matrix A

$$det(A) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$\begin{aligned} &= 1(1 \times 1 - 0 \times 1) - 1(0 \times 1 - 1 \times 1) + 0(0 \times 0 - 1 \times 1) \\ &= 1 + 1 + 0 = 2 \neq 0 \end{aligned}$$

Since:

$$\det\ (A) \neq 0$$

the columns are linearly independent and generate $\mathbb{R}^3$

Thus:

$$Im(f) = \mathbb{R}^3$$

4. Is f an isomorphism?.....(1pts)

A linear application is an isomorphism if it is bijective.

Since:

$\ker(f) = \{0_{\mathbb{R}^3}\} \Rightarrow$ f is injective.

And:

$Im(f) = \mathbb{R}^3 \Rightarrow f$ is surjective.

Therefore, f is an isomorphism.

Solution of Exercise 3

1. Proof that $A^{-1}BA = B$ .....(3pts)Since $AB = BA$, multiply both sides on the left by A^{-1}

$$A^{-1}AB = A^{-1}BA$$

This gives:

$$B = A^{-1}BA.$$

2. General form of A^n .....(3pts)

We compute the first powers:

$A = A^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, $A^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$. We observe the pattern:

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

Proof by induction

Base case ($n = 1$):

$A^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is true.

Inductive step:

Assume:

$A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ Then:

$A^{k+1} = A^k A = A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$

The formula holds for $k + 1$

Conclusion

$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, for all $n \in \mathbb{N}$.

Model Exam 2

? Exercise 1 (5 points)

1. Give the definition of a vector subspace of a vector space over the field $\mathbb{R}$.
2. Define what it means for a set of vectors to be:
   a) linearly independent.
   b) a basis of a vector space.
3. State the relationship between:
   a) dimension
   b) basis of a finite-dimensional vector space
4. State a condition under which a square matrix is invertible.
5. Define the rank of a matrix.

? Exercise 2 (7 points)

Let

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$

defined by:

$$f\ (x, y, z) = (x - y + z,\ \ 2x + y - z)$$

1. Verify that f is a linear application.
2. Determine the kernel of f , $(\ker(f))$
3. Determine the image of f , $(\mathrm{Im}(f))$
4. Compute: **a)** the dimension of the kernel
   b) the dimension of the image.
5. Determine the rank of the linear application f.

? Exercise 3 (8 points)

Let the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

1. Compute: A^2
2. Compute: $\det(A)$
3. If A is invertible, find $\mathbf{A}^{-1}$.
4. Determine the rank of the matrix A.

The Model Solution for the 2 Exam

Solution of Exercise 1

1. Definition of a vector subspace.....(1.5)

   A nonempty subset F of a vector space E over the field $\mathbb{R}$ is called a vector subspace of E if it satisfies the following conditions:

   - $0_E \in F$
   - For all U, $V \in F$, we have $U + V \in F$ (closed under addition).
   - For all $\lambda \in \mathbb{R}$ and For all $U \in F$, we have $\lambda U \in F$ (closed under scalar multiplication)

2. .....(1.5pt)

   a. Linearly independent

   A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in a vector space E is said to be linearly independent if:

   For all $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{R}$,

   $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$ implies $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

   b. Basis of a vector space

   A family of vectors $\{v_1, v_2, \ldots, v_n\}$ is called a basis of a vector space E if:

   - the vectors are linearly independent, and
   - they generate the space E (that is, every vector of E can be written as a linear combination of them).

3. Relationship between dimension and basis.....(1pt)

   In a finite-dimensional vector space:

   - All bases have the same number of vectors.
   - This number is called the dimension of the vector space.

   In other words:

   The dimension of a finite-dimensional vector space is equal to the number of vectors in any basis of that space.

4. Condition for a square matrix to be invertible Asquare matrix

   A is invertible if and only if: $\det(A) \neq 0$.....(0.5pt)

   (Equivalently, the rank of A is equal to its size).

   i.e. if a square matrix A of order $(n \times n)$ then :

   $$A \text{ is invertible.} \Leftrightarrow \det(A) \neq 0. \Leftrightarrow rank(A) = n$$

5. Definition of the rank of a matrix .....(0.5pt)

   The rank of a matrix A is the dimension of the vector space generated by its rows (or columns).

   Equivalently:

   The rank of a matrix is the maximum number of linearly independent rows (or columns) of the matrix.

Solution of Exercise 2

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$

$$f\ (x, y, z) = (x - y + z,\ \ 2x + y - z)$$

Show that f is a linear application.....(2pts)

We verify linearity:

For all $U=(x_1,y_1,z_1)$, $V=(x_2,y_2,z_2)$, and for all $\lambda\in\mathbb{R}$

- $f(U+V)=f((x_1,y_1,z_1)+(x_2,y_2,z_2))$
$=f((x_1+x_2,\ y_1+y_2,\ z_1+z_2))$
$=(x_1+x_2-y_1-y_2+z_1+z_2,\ \ 2x_1+2x_2+y_1+y_2-z_1-z_2)$
$=((x_1-y_1+z_1)+(x_2-y_2+z_2),,\ \ (2x_1+y_1-z_1)+(2x_2+y_2-z_2)))$
$=((x_1-y_1+z_1),\ (2x_1+y_1-z_1))+(\ (x_2-y_2+z_2),\ (2x_2+y_2-z_2))$
$=f(x_1,y_1,z_1)+f(x_2,y_2,z_2)$
$=f(U)+f(V).$
- $f(\lambda U)=f(\lambda(x_1,y_1,z_1))=(\lambda x_1-\lambda y_1+\lambda z_1,\ 2\lambda x_1+\lambda y_1-\lambda z_1)$
$=\lambda f(U)$

Therefore, f is a linear application.

2. Kernel of f.....(1pt)

$ker(f)=\{(x,y,z)\in\mathbb{R}^3/\quad f(x,y,z)=0_{\mathbb{R}^2}\}$
$=\{(x,y,z)\in\mathbb{R}^3/\quad (x-y+z,\ \ 2x+y-z)=(0,0)\}$

So,

$$\begin{cases} x-y+z=0\\ 2x+y-z=0\end{cases}$$

Add equations:

$x-y+z+2x+y-z=0\Rightarrow x=0$

Substitute

$-y+z=0\Rightarrow y=z$

So,

$ker(f)=\{(0,y,y)\ \ /y\in\mathbb{R}\}=\{y(0,1,1)\ \ /y\in\mathbb{R}\}$

3. Image of f.....(1pt)

$Im(f)=\{f(x,y,z)\ \ /\ \ (x,y,z)\in\mathbb{R}^3\}$
$=\{(x-y+z,\ \ 2x+y-z)\ \ /\ \ (x,y,z)\in\mathbb{R}^3\}$
$=\{x(1,2)+y(-1,1)+z(1,-1)\ \ /\ \ (x,y,z)\in\mathbb{R}^3\}$

Tow vectors are clearly linearly independent.

4. $dim(ker(f))=1$.....(1pt)

$dim(Im(f))=2$.....**(1pt)**

5. $rank(f)=dim(Imf)=2$.....(1pt)

Solution of Exercise 3

$$A=\begin{pmatrix}1&1&0\\2&1&1\\1&0&1\end{pmatrix}$$

Compute A^2.....(2pt)

$$A^2=A\times A$$

Compute row by column:

$$A^2 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Row 1:

- 1(1)+1(2)+0(1)=3
- 1(1)+1(1)+0(0)=2
- 1(0)+1(1)+0(1)=1

Row 2:

- 2(1)+1(2)+1(1)=5
- 2(1)+1(1)+1(0)=3
- 2(0)+1(1)+1(1)=2

Row 3:

- 1(1)+0(2)+1(1)=2
- 1(1)+0(1)+1(0)=1
- 1(0)+0(1)+1(1)=1

So,

$$A^2 = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

2. Compute $\det(A)$.....(2pt)

$$\det(A) = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

Expand along first row:

$$\det(A) = 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}$$

So,

$$\det(A) = 1(1(1) - 0(1)) - 1(2(1) - 1(1)) + 0(2(0) - 1(1))$$

Then,

$$\det(A) = 1(1) - 1(1) + 0(-1)$$

Therfore,

$$\det(A) = 0$$

3. .....(2pt)

$\det(A) = 0 \Rightarrow$ A is NOT invertible

So the matrix has no inverse

Then

A^{-1} does not exist.

4. .....(2pt)

Since determinant of A is zero then $rank\ (A) < 3$.

Rows are not multiples, and two rows are independent.

So,

$$rank\ (A) = 2$$

Model Exam 3

? Exercise 1 (6 points)

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (2x - y \ , \ x + 3y).$$

1. Show that f is a linear application.
2. Compute: $f\ (1, 2)$, $f\ (-1, 3)$.
3. Determine the kernel of f, denoted by $Ker(f)$

? Exercise 2 (6 points)

In the vector space $\mathbb{R}^3$, consider the vectors

$$u = (1, 0, 2), \quad v = (2, 1, 3), \quad w = (0, 1, 1).$$

1. Verify whether the vectors u, v, w are linearly independent.
2. Do the vectors u, v, w form a basis of $\mathbb{R}^3$?
3. Express the vector $a = (3, 2, 5)$ as a linear combination of the vectors u, v, w.

? Exercise 3 (8 points)

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & a & 1 \\ 1 & 3 & a \end{pmatrix}, \qquad a \in \mathbb{R}$$

1. Compute the determinant $\det(A)$ as a function of a
2. Deduce the values of a for which the matrix A is invertible.
3. Discuss the rank of the matrix A according to the values of a
4. Solve the following linear system using Cramer's rule whenever possible:

$$\begin{cases} x + 2y + z = 1, \\ 2x + ay + 3z = 2, \\ x + y + az = 1. \end{cases}$$

5. Discuss the number of solutions of the previous system according to the values of the parameter a.

The Model Solution for the 3 Exam

Solution of Exercise 1

1. Proof that f is linear

   A map f is linear if, for all vectors $u, v \in \mathbb{R}^2$, and all scalars $\lambda \in \mathbb{R}$

   $$f(u+v) = f(u) + f(v)$$

   and

   $$f(\lambda u) = \lambda f(u).$$

   Let

   $u = (x_1, y_1), \quad v = (x_2, y_2).$

   Then

   $u + v = (x_1 + x_2,\ y_1 + y_2).$

   We compute:

   $$\begin{aligned} f(u+v) = f(x_1+x_2,\ y_1+y_2) &= (2(x_1+x_2) - (y_1+y_2), (x_1+x_2) + 3(y_1+y_2)) \\ &= (2x_1 - y_1, x_1 + 3y_1) + (2x_2 - y_2, x_2 + 3y_2) \\ &= f\ (x_1, y_1) + f\ (x_2, y_2) \\ &= f(u) + f(v) \quad \text{.....(1pt)} \end{aligned}$$

   Now let $\lambda \in \mathbb{R}$ We have:

   $$\begin{aligned} f(\lambda u) = f(\lambda(x_1, y_1)) = f\ (\lambda x_1, \lambda y_1) &= (2\lambda x_1 - \lambda y_1,\ \ \lambda x_1 + 3\lambda y_1) \\ &= \lambda(2x_1 - y_1,\ \ x_1 + 3y_1) \\ &= \lambda f\ (x_1, y_1) \\ &= \lambda f\ (u) \quad \text{.....(1pt)} \end{aligned}$$

   Hence,

   $$f \text{ is a linear application.}$$

2. Computation of $f\ (1,2)$ and $f\ (-1,3)$

   First vector

   $f\ (1,2) = (2(1)\ -2\ \ ,\ \ 1+3(2)) = (0,7)$ .....(1pt)

   Thus,

   $$f(1,2) = (0,7)$$

   Second vector

   $f\ (-1,3) = (2(-1) - 3\ ,\ \ -1+3(3)) = (-5,8)$.....(1pt)

   Thus,

   $$f(-1,3) = (-5,8)$$

3. Kernel of f is the set of vectors (x,y) such that

   $f(x,y) = (0,0).$

   $(2x\ -y\ \ ,\ \ x+3y) = (0,0)$ Thus,

   $$\begin{cases} 2x - y = 0 \\ x + 3y = 0 \end{cases}$$

From the first equation:

$$y = 2x.$$

Substituting into the second equation:

$x + 3(2x) = 0$

thus

$$x = 0.$$

Hence,

$$y = 0$$

Therefore,

$$\ker(f) = \{(0,0)\} \text{.....(2pts)}$$

Solution of Exercise 2

1. Linear independence of u, v, w .....(2pts)

   $\forall \lambda_1,\ \lambda_2\ ,\ \lambda_3 \in \mathbb{R} : \lambda_1 u + \lambda_2\ v + \lambda_3\ w = 0_{\mathbb{R}^3} \overset{?}{\Rightarrow} \lambda_1 = \lambda_2 = \lambda_3 = 0$

   $\lambda_1 u + \lambda_2\ v + \lambda_3\ w = 0_{\mathbb{R}^3} \Leftrightarrow \lambda_1(1,0,2) + \lambda_2\ (2,1,3) + \lambda_3\ (0,1,1) = (0,0,0)$

   $$\Leftrightarrow (\lambda_1 + 2\lambda_2,\ \lambda_2 + \lambda_3, 2\lambda_1 + 3\lambda_2 + \lambda_3) = (0,0,0)$$

   This gives the system:

   $$\begin{cases} \lambda_1 + 2\lambda_2 = 0 \\ \lambda_2 + \lambda_3 = 0 \\ 2\lambda_1 + 3\lambda_2 + \lambda_3 = 0 \end{cases}$$

   From the second equation:

   $$\lambda_3 = -\lambda_2.$$

   From the first equation:

   $$\lambda_1 = -2\lambda_2$$

   Substituting into the third equation:

   $$\begin{aligned} 2(-2\lambda_2) + 3\lambda_2 - \lambda_2 &= 0 \\ -2\lambda_2 &= 0 \\ \lambda_2 &= 0 \end{aligned}$$

   Hence,

   $$\lambda_1 = 0, \quad \lambda_3 = 0.$$

   Therefore,

   $$u, v, w \text{ are linearly independent}$$

2. Basis of $\mathbb{R}^3$.....(2pts)

   Since u, v, w are three linearly independent vectors in $\mathbb{R}^3$, they form a basis of $\mathbb{R}^3$

3. Writing a as a linear combination of u, v, w .....(2pts)

We seek scalars α_1,α_2,α_3 such that

$a = \alpha_1 u + \alpha_2 v + \alpha_3 w$

That is,

$(3, 2, 5) = \alpha_1(1, 0, 2) + \alpha_2(2, 1, 3) + \alpha_3(0, 1, 1)$

$(3, 2, 5) = (\alpha_1 + 2\alpha_2, \alpha_2 + \alpha_3, 2\alpha_1 + 3\alpha_2 + \alpha_3)$

This gives the system:

$$\begin{cases} \alpha_1 + 2\alpha_2 = 3 \\ \alpha_2 + \alpha_3 = 2 \\ 2\alpha_1 + 3\alpha_2 + \alpha_3 = 5 \end{cases}$$

From the second equation:

$$\alpha_3 = 2 - \alpha_2$$

From the first equation:

$$\alpha_1 = 3 - 2\alpha_2$$

Substituting into the third equation:

$2(3 - 2\alpha_2) + 3\alpha_2 + (2 - \alpha_2) = 5$

$-4\alpha_2 + 3\alpha_2 - \alpha_2 = 5 - 6 - 2$

$-2\alpha_2 = -3$

$$\alpha_2 = \frac{3}{2}$$

Then,

$\alpha_3 = 2 - \frac{3}{2} = \frac{4-3}{2}$

$$\alpha_3 = \frac{1}{2}$$

$\alpha_1 = 3 - 2\alpha_2 = 3 - 2\frac{3}{2} = 0$

$$\alpha_1 = 0$$

Therefore,

$$a = 0u + \frac{3}{2}v + \frac{1}{2}w$$

or equivalently,

$$a = \frac{3}{2}v + \frac{1}{2}w$$

Solution of Exercise 3

1. Computation of the determinant .....(2pts)

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & a & 1 \\ 1 & 3 & a \end{vmatrix}$$

Developing along the first row:

$$\det(A) = 1\begin{vmatrix} a & 1 \\ 3 & a \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & a \end{vmatrix} + 1\begin{vmatrix} 2 & a \\ 1 & 3 \end{vmatrix}$$
$$= 1(a^2 - 3) - 2(2a - 1) + 1(6 - a)$$
$$= a^2 - 3 - 4a + 2 + 6 - a$$
$$= a^2 - 5a + 5$$

2. Values of a for which A is invertible.....(2pts)

A matrix is invertible if and only if its determinant is nonzero.

$a^2 - 5a + 5 \neq 0$

Solving $a^2 - 5a + 5 = 0$

$a = \frac{5 \pm \sqrt{25-20}}{2} = \frac{5 \pm \sqrt{5}}{2}$

Hence,

$$A \text{ is invertible for } \quad a \neq \frac{5+\sqrt{5}}{2} \ , \quad a \neq \frac{5-\sqrt{5}}{2}$$

3. Rank of the matrix A.....(2pts)

• If $a \neq \frac{5 \pm \sqrt{5}}{2}$

Then,

$$det(A) \neq 0$$

Therefore,

$$rg(A) = 3$$

• If $a = \frac{5 \pm \sqrt{5}}{2}$

Then,

$$det(A) = 0$$

So the rank is strictly less than 3 ($rg(A) < 3$)

Since at least one 2×2 minor is nonzero, for example

$$\begin{vmatrix} 1 & 2 \\ 2 & a \end{vmatrix} = a - 4$$

which is nonzero for these values of a, we obtain for $a = \frac{5 \pm \sqrt{5}}{2}$

$$rg(A) = 2$$

4) Resolution of the system using Cramer's rule.....(2pts)

The system is

$$\begin{cases} x + 2y + z = 1, \\ 2x + ay + 3z = 2, \\ x + y + az = 1. \end{cases} \Leftrightarrow SX = B$$

With $S = \begin{pmatrix} 1 & 2 & 1 \\ 2 & a & 3 \\ 1 & 1 & a \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & a & 3 \\ 1 & 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

The determinant of the system is: $\det(S) = a^2 - 5a + 5$

For

$\det(S) \neq 0$,

the system has a unique solution.

$$x = \frac{\det(S_x)}{\det(S)} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & a & 3 \\ 1 & 1 & a \end{vmatrix}}{a^2-5a+5} = \frac{a^2-5a+5}{a^2-5a+5} = 1$$

$$y = \frac{\det(S_y)}{\det(S)} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & a \end{vmatrix}}{a^2-5a+5} = \frac{a-1}{a^2-5a+5}$$

$$z = \frac{\det(S_y)}{\det(S)} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & a & 2 \\ 1 & 1 & 1 \end{vmatrix}}{a^2-5a+5} = \frac{2-a}{a^2-5a+5}$$

Therefore, for

$a \neq \frac{5\pm\sqrt{5}}{2}$,

the unique solution is

$$(x, y, z) = (1, \ \frac{a-1}{a^2-5a+5}, \ \frac{2-a}{a^2-5a+5})$$

5) Discussion of the number of solutions

• If $a \neq \frac{5\pm\sqrt{5}}{2}$

Then

$\det(S) \neq 0$,

hence the system admits a unique solution.

• If $a = \frac{5\pm\sqrt{5}}{2}$

Then

$\det(S) = 0$,

Cramer's rule cannot be applied.

In this case, one studies the ranks of the coefficient matrix and the augmented matrix to determine whether the system has infinitely many solutions or no solution.

Model Exam 4

? Exercise 1 (5 points)

Show that $E = \{(x + y + z,\ x - y,\ z) /\ \ x, y, z \in \mathbb{R}^3\}$ is a vector subspace of $\mathbb{R}^3$

? Exercise 2 (7.5 points)

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x - y, -3x + 3y)$$

1. Show that f is a linear application.
2. Give a basis of $Ker(f)$ and a basis of $Im(f)$.
3. Show that f is neither injective no surjective.

? Exercise 3 (7.5 points)

Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear application whose matrix in the canonical basis of $\mathbb{R}^4$ and $\mathbb{R}^3$ is

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & -2 & 5 & -11 \end{pmatrix}$$

1. Determine a basis of the $Ker(f\ )$.
2. Determine a basis of the $Im(f)$. What is the rank of f?

The Model Solution for the 4 Exam

Solution of Exercise 1

- $0_{\mathbb{R}^3} = (0,0,0) \in E$ because $(0,0,0) = (0+0+0, 0-0, 0))$
  So $E \neq \phi$ .....**(2 pts)**

- $\forall X, Y \in E$, λ, $\mu \in \mathbb{R}$ show that $\lambda X + \mu Y \overset{?}{\in} E$; on a :
  $X \in E, \exists (x,y,z) \in \mathbb{R}^3 / X = (x+y+z, x-y, z)$,
  $Y \in E$,$\exists (x^{'}, y^{'}, z^{'}) \in \mathbb{R}^3 / X = (x^{'}+y^{'}+z^{'}, x^{'}-y^{'}, z^{'})$,
  $\lambda X + \mu Y = \lambda(x+y+z, x-y, z) + \mu(x^{'}+y^{'}+z^{'}, x^{'}-y^{'}, z^{'})$
  $= (\lambda x + \lambda y + \lambda z, \lambda x - \lambda y, \lambda z) + (\mu x^{'} + \mu y^{'} + \mu z^{'}, \mu x^{'} - \mu y^{'}, \mu z^{'})$
  $= ((\lambda x + \mu x^{'}) + (\lambda y + \mu y^{'}) + (\lambda z + \mu z^{'}), (\lambda x + \mu x^{'}) - (\lambda y + \mu y^{'}), (\lambda z + \mu z^{'}))$
  from where $\exists x^{"} = \lambda x + \mu x^{'} \in \mathbb{R}$, $\exists y^{"} = \lambda y + \mu y^{'} \in \mathbb{R}$, and $\exists z^{"} = \lambda z + \mu z^{'} \in \mathbb{R}$
  So, $\lambda X + \mu Y \in E$ .....**(3 pts)**
  Then E is a vector subspace of $\mathbb{R}^3$

Solution of Exercise 2

1. Let $(x,y), (x^{'}, y^{'}) \in \mathbb{R}^2$, $\forall \lambda$, $\mu \in \mathbb{R}$,
   $f(\lambda(x,y) + \mu(x^{'}, y^{'})) = f(\lambda x + \mu x^{'}, \lambda y + \mu y^{'})$
   $= (\lambda x + \mu x^{'} - \lambda y - \mu y^{'}, -3(\lambda x + \mu x^{'}) + 3(\lambda y + \mu y^{'}))$
   $= (\lambda(x-y) + \mu(x^{'} - y^{'}), \lambda(-3x+3y) + \mu(-3x^{'} + 3y^{'}))$
   $= (\lambda(x-y), \lambda(-3x+3y)) + (\mu(x^{'} - y^{'}), \mu(-3x^{'} + 3y^{'}))$
   $= \lambda((x-y), (-3x+3y)) + \mu((x^{'} - y^{'}), (-3x^{'} + 3y^{'}))$
   $= \lambda f((x,y) + \mu f(x^{'}, y^{'})$
   So, f is a linear application.....**(2 pts)**

2. $f(1,1) = (0,0) = f(0,0)$ and yet $(1,1) \neq (0,0)$
   so f is not injective.....**(1.5 pt)**
   We will show that $(1,0)$ has no antecedent. Suppose there exists (x,y)
   such that $(1,0) = f(x,y) \Leftrightarrow (1,0) = (x-y, -3x+3y)$
   $$\Leftrightarrow \begin{cases} 1 = x - y \\ 0 = -3x + 3y \end{cases}$$
   $$\Leftrightarrow \begin{cases} 1 = x - y \\ x = y \end{cases}$$
   $$\Leftrightarrow \begin{cases} 1 = 0 \\ x = y \end{cases} \text{ it's impossible.}$$
   so f is not surjective.....**(1.5 pt)**
   f is an endomorphism so f is injective if and only if f is surjective. Here, f is not injective so f is not surjective.

3. $(x,y) \in ker(f) \Leftrightarrow (x-y, -3x+3y) = (0,0) \Leftrightarrow \begin{cases} 0 = x - y \\ 0 = -3x + 3y \end{cases}$
   $\Leftrightarrow x = y$
   So $(x,x) = x(1,1)$, $(1,1)$ is a non-zero vector which generates $ker(f)$, it is a basis of $\ker(f)$
   $f(e_1) = (1-0, -3 \times 1 + 3 \times 0) = (1,-3) = e_1 - 3e_2$ and
   $f(e_2) = ((0-1, -3 \times 0 + 3 \times 1) = (-1,3) = -e_1 + 3e_2$

$Im(f) = Vect(f(e_1), f(e_2)) = Vect(e_1 - 3e_2, -e_1 + 3e_2) = Vect(e_1 - 3e_2)$

$e_1 - 3e_2$ is a non-zero vector that generates $Im(f)$,

it is a base of $Im(f)$.....**(2.5 pts)**

Solution of Exercise 3

1. Let $X \in \mathbb{R}^4$, So $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

$$X \in Ker(f) \Leftrightarrow AX = O_{\mathbb{R}^3} \Leftrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & -2 & 5 & -11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 + x_3 + 3x_4 = 0 \\ x_1 + x_2 + 2x_3 + x_4 = 0 \\ x_1 - 2x_2 + 5x_3 - 11x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -3x_3 \\ x_2 = x_3 \\ x_4 = 0 \end{cases}$$

$X = (-3x_3, x_3, x_3, 0) = x_3(-3, 1, 1, 0)$

Then $(-3, 1, 1, 0)$ is a basis of $ker(f\ )$, so $dim\ ker(f\ ) = 1$.....**(3 pts)**

2. According to the rank theorem

$dim\ ker(f\ ) + dim\ Im(f\ ) = dim\ \mathbb{R}^4 = 4$

So, $dim\ Im(f\ ) = 3 = .dim\ \mathbb{R}^3$

Then $Im(f\ ) = \mathbb{R}^3$ and $rg(A) = 3$.....**(3 pts)**

Model Exam 5

? Exercise 1 (8 points)

Let

$f_1 : \mathbb{R}^3 \to \mathbb{R}$, $f_2 : \mathbb{R}^3 \to \mathbb{R}$

$(x, y, z) \mapsto f_1(x, y, z) = x - y$, $(x, y, z) \mapsto f_2(x, y, z) = y - z$

and

$$\begin{array}{ccc} f & : & \mathbb{R}^3 \to \mathbb{R}^3 \\ (x, y, z) & \mapsto & f(x, y, z) = (f_1(x, y, z), \quad f_2(x, y, z), \quad z) \end{array}$$

and let $E = \left\{(x, y, z) \in \mathbb{R}^3 : f_1(x, y, z) = 0 \ \wedge f_2(x, y, z) = 0\right\}$

1. Show that E is a subspace of $\mathbb{R}^3$.
2. Give a basis for E, and its dimension.
3. Show that f is a linear application
4. Give a basis and the dimension of $ker\ f$
5. Give a basis and the dimension of $Im\ f$
6. is f bijective?

? Exercise 2 (6 points)

Let

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

1. Calculate A^2.
2. Find the real numbers α, β, γ such that.$\alpha A^2 + \beta A + \gamma I..= 0$.
   I: Unit matrix, 0: Zero matrix.
3. From a previous relation deduce that A is invertible and determine its inverse.

? Exercise 3 (6 points)

Let the following matrix:

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}, \text{ and } \quad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Find the solution of the linear sentence using Cramer's method $AX = B$.

The Model Solution for the 5 Exam

Solution of Exercise 1

1. Prove that E is a vector subspace

   $(0,0,0) \in E$ because $(0-0=0 \ \wedge \ 0-0=0)$

   $\forall (x,y,z),(x\prime,y\prime,z\prime) \in E;\ \forall \alpha,\beta \in \mathbb{R}: \alpha(x,y,z)+\beta(x\prime,y\prime,z\prime) \overset{?}{\in} E$

   $(\alpha x,\alpha y,\alpha z)+(\beta x\prime,\beta y\prime,\beta z\prime) \overset{?}{\in} E \Leftrightarrow (\alpha x+\beta x\prime, \quad \alpha y+\beta y\prime, \quad \alpha z+\beta z\prime) \overset{?}{\in} E$

   $$\Leftrightarrow \begin{cases} \alpha x+\beta x\prime-\alpha y+\beta y\prime \overset{?}{=} 0 \\ \alpha y+\beta y\prime-\alpha z+\beta z\prime \overset{?}{=} 0 \end{cases}$$

   $$(x,y,z) \in E \Leftrightarrow \begin{cases} x-y=0 \\ y-z=0 \end{cases} \Leftrightarrow \begin{cases} \alpha x-\alpha y=0.....(1) \\ \alpha y-\alpha z=0.....(2) \end{cases} \text{ with } \alpha \in \mathbb{R}$$

   $$(x\prime,y\prime,z\prime) \in E \Leftrightarrow \begin{cases} x^{'}-y^{'}=0 \\ y^{'}-z^{'}=0 \end{cases} \Leftrightarrow \begin{cases} \beta x^{'}-\beta y^{'}=0.....(3) \\ \beta y^{'}-\beta z^{'}=0.....(4) \end{cases} \text{ with } \beta \in \mathbb{R}$$

   By adding (1) with (3) and (2) with (4), we find what is required,
   and we conclude that E is a vector subspace of space $\mathbb{R}^3$
   on the field $\mathbb{R}$

2. Find the basis and dimension of the subspace E

   $$(x,y,z) \in E \Leftrightarrow \begin{cases} x-y=0 \\ y-z=0 \end{cases} \Leftrightarrow \begin{cases} x=y \\ y=z \end{cases}$$

   $(x,y,z) \in E \Leftrightarrow (x,y,z)=(x,x,x)=x(1,1,1)$
   So, $v=(1,1,1)$ is a basis of E Then $dimE=1$.

3. Prove that f is a linear application

   $\forall (x,y,z),(x\prime,y\prime,z\prime) \in E;\ \forall \alpha,\beta \in \mathbb{R}: f(\alpha(x,y,z)+\beta(x\prime,y\prime,z\prime))=\alpha f(x,y,z)+\beta f(x\prime,y\prime,z\prime)$

   $$\begin{aligned} f(\alpha(x,y,z)+\beta(x\prime,y\prime,z\prime)) &= f(\alpha x+\beta x\prime,\alpha y+\beta y\prime,\alpha z+\beta z\prime) \\ &= (\alpha x+\beta x\prime-\alpha y-\beta y\prime, \ \alpha y+\beta y\prime-\alpha z-\beta z\prime, \ \alpha z+\beta z\prime) \\ &= (\alpha x-\alpha y,\alpha y-\alpha z,\alpha z)+(\beta x\prime-\beta y\prime,\beta y\prime-\alpha z-\beta z\prime,\beta z) \\ &= \alpha f(x,y,z)+\beta f(x\prime,y\prime,z\prime) \end{aligned}$$

   So f is linear application.

4. $Ker f=\{(x,y,z) \in \mathbb{R}^3 : f((x,y,z)=(0,0,0)\}$

   $$f((x,y,z)=(0,0,0) \Leftrightarrow \begin{cases} x-y=0 \\ y-z=0 \\ z=0 \end{cases} \Leftrightarrow x=y=z=0$$

   $Ker f=\{(0,0,0)\} \Rightarrow \dim \ker f=0 \Rightarrow f$ is injective
   Since the starting set is the same as the arrival set, the application is surjective,
   i.e., f is bijective, and therefore
   $Im(f)=\mathbb{R}^3 \Rightarrow \dim Im(f)=3$.

Solution of Exercise 2

$$A=\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

1. $A^2 = A \times A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$
$= \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix}$

2. $\alpha A^2 + \beta A + \gamma I_{..} = 0. \Leftrightarrow \alpha \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix} + \beta \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 8\alpha+\gamma & 4\alpha+2\beta & 4\alpha+2\beta \\ 4\alpha+2\beta & 8\alpha+\gamma & 4\alpha+2\beta \\ 4\alpha+2\beta & 4\alpha+2\beta & 8\alpha+\gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{cases} 8\alpha+\gamma = 0 \\ 2\alpha+\beta = 0 \end{cases} \Leftrightarrow \begin{cases} -8\alpha = \gamma \\ -2\alpha = \beta \end{cases}$

by taking $\alpha = 1$, We find $\gamma = -8$ and $\beta = -2$.

In the end we find that: $\alpha A^2 + \beta A + \gamma I_{..} = 0$.

Solution of Exercise 3

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}, \text{ and } \quad B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$detA = -7 \neq 0$,
$rgA = n = p = 3$ (is a cramer system).

$$x = \frac{detA_1}{detA} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix}}{-7} = \frac{-9}{-7} = \frac{9}{7}.$$

$$y = \frac{detA_2}{detA} = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 3 & 3 & 1 \end{vmatrix}}{-7} = \frac{5}{-7}.$$

$$z = \frac{detA_2}{detA} = \frac{\begin{vmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{vmatrix}}{-7} = \frac{1}{-7}.$$

So, the solution is:

$X = (\frac{9}{7}, \frac{-5}{7}, \frac{-1}{7})$

Model Exam 6

? Exercise 1 (8 points)

Let

$$\begin{array}{rcl} f & : & \mathbb{R}^3 \to \mathbb{R}^3 \\ (x,y,z) & \mapsto & f(x,y,z) = (-2x+y+z,\ \ x-2y+z,\ x+y\ -2z) \end{array}$$

1. Show that f is a linear application
2. Give a basis and the dimension of $ker\ f$
3. Give the dimension of $Im\ f$
4. Is f bijective?

? Exercise 2 (6 points)

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

1. Calculate A^2.
2. Find the real numbers α, β such that.$A^2 + \alpha A + \beta I = 0$.
   I: Identity matrix, 0: Zero matrix.
3. From a previous relation deduce that A is invertible and determine its inverse.

? Exercise 3 (6 points)

Let the following matrix:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & -1 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$$

Find the solution X of the linear system $AX = B$ using Cramer's Rule.

The Model Solution for the 6 Exam

Solution of Exercise 1

1. Prove that f is a linear application

$\forall (x,y,z),(x\prime,y\prime,z\prime) \in E;\ \forall \alpha,\beta \in \mathbb{R}: f(\alpha(x,y,z)+\beta(x\prime,y\prime,z\prime)) = \alpha f(x,y,z)+\beta f(x\prime,y\prime,z\prime)$

$$\begin{aligned} f(\alpha(x,y,z)+\beta(x\prime,y\prime,z\prime)) &= f(\alpha x+\beta x\prime, \alpha y+\beta y\prime, \alpha z+\beta z\prime) \\ &= (-2\alpha x-2\beta x\prime+\alpha y+\beta y\prime+\alpha z+\beta z\prime,\ \alpha y+\beta y\prime-2\alpha z- \\ &\quad 2\beta z\prime+\alpha z+\beta z\prime,\ \alpha x+\beta x\prime+\alpha y+\beta y\prime-2\alpha z-2\beta z\prime) \\ &= (-2\alpha x+\alpha y+\alpha z,\ \alpha y-2\alpha z+\alpha z,\ \alpha x+\alpha y-2\alpha z)+ \\ &\quad (-2\beta x\prime+\beta y\prime+\beta z\prime,\ \beta y\prime-2\beta z\prime+\beta z\prime,\ \beta x\prime+\beta y\prime-2\beta z\prime) \\ &= \alpha f(x,y,z)+\beta f(x\prime,y\prime,z\prime) \end{aligned}$$

So f is linear application..... **(2pts)**

2. $Ker f = \{(x,y,z) \in \mathbb{R}^3 : f((x,y,z) = (0,0,0)\}$

$$f((x,y,z)=(0,0,0) \Leftrightarrow \begin{cases} -2x+y+z=0 \\ x-2y+z=0 \\ x+y-2z=0 \end{cases} \Leftrightarrow x=y=z$$

$Ker f = \{(1,1,1)\} \Rightarrow \dim \ker f = 1 \Rightarrow f$ is not injective..... **(2pts)**

3. $\dim \ker f + \dim Im(f) = \dim \mathbb{R}^3 = 3$

$\dim Im(f) = 3-1 = 2 \Rightarrow$ is not surjective..... **(2pts)**

4. Then f is not bijective because f is not injective and not surjective..... **(2pts)**

Solution of Exercise 2

1. We are given:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

2. Compute A^2

$$A^2 = A \times A$$

$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Compute entries:

- $(1,1) : 2\cdot 2+1\cdot 1 = 4+1 = 5$
- $(1,2) : 2\cdot 1+1\cdot 2 = 2+2 = 4$
- $(2,1) : 1\cdot 2+2\cdot 1 = 2+2 = 4$
- $(2,2) : 1\cdot 1+2\cdot 2 = 1+4 = 5$

So,

$$A^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

3. Find α, β such that

$$A^2 + \alpha A + \beta I = 0$$

We write each matrix:

$$A^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}, \quad \alpha A = \begin{pmatrix} 2\alpha & \alpha \\ \alpha & 2\alpha \end{pmatrix}, \quad \beta I = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}$$

Now sum:

$$A^2 + \alpha A + \beta I = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 2\alpha & \alpha \\ \alpha & 2\alpha \end{pmatrix} + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This must equal zero matrix:

$$A^2 + \alpha A + \beta I = \begin{pmatrix} 5 + 2\alpha + \beta & 4 + \alpha \\ 4 + \alpha & 5 + 2\alpha + \beta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So we solve:
Off-diagonal:

$$4 + \alpha = 0 \Rightarrow \alpha = -4$$

Diagonal:

$$5 + 2(-4) + \beta = 0$$

$$5 - 8 + \beta = 0 \Rightarrow \beta = 3$$

Then,

$$\alpha = -4, \qquad \beta = 3$$

4. Deduce that
A is invertible and find A^{-1}
From:

$$A^2 - 4A + 3I = 0$$

Rearrange:

$$A2 - 4A = -3I$$

Factor:

$$A(A - 4I) = -3I$$

Multiply both sides by $\frac{-1}{3}$

$$A \cdot (\frac{-1}{3}(A - 4I)) = I$$

So:

$$A^{-1} = \frac{-1}{3}(A - 4I)$$

Compute $A - 4I$

$$A-4I=\begin{pmatrix}2&1\\1&2\end{pmatrix}-\begin{pmatrix}4&0\\0&4\end{pmatrix}=\begin{pmatrix}-2&1\\1&-2\end{pmatrix}$$

So,

$$A^{-1}=\frac{-1}{3}\begin{pmatrix}-2&1\\1&-2\end{pmatrix}=\begin{pmatrix}\frac{2}{3}&\frac{-1}{3}\\\frac{-1}{3}&\frac{2}{3}\end{pmatrix}$$

Solution of Exercise 3

Let

$$A=\begin{pmatrix}1&-1&2\\2&1&1\\3&2&-1\end{pmatrix},\qquad B=\begin{pmatrix}4\\5\\3\end{pmatrix}$$

Find the solution X of the linear system

$$AX=B$$

Using Cramer's Rule.
Step 1: Compute det(A)
Expanding along the first row:

$$\begin{aligned}\det(A)&=1\cdot\begin{vmatrix}1&1\\2&-1\end{vmatrix}-(-1)\cdot\begin{vmatrix}2&1\\3&-1\end{vmatrix}+2\cdot\begin{vmatrix}2&1\\3&2\end{vmatrix}\\&=1\cdot(1\times(-1)-2\times1)+1\cdot(2\times(-1)-3\times1)+2\cdot(2\times2-3\times1)\\&=-3-5+2\\&=-6\end{aligned}$$

Thus,

$$\det(A)=-6\neq0$$

Since $\det(A)\neq0$, the system has a unique solution.
Step 2: Compute
Replace the first column of A by B

$$A_x=\begin{pmatrix}4&-1&2\\5&1&1\\3&2&-1\end{pmatrix}$$

$$\det(A_x)=4\cdot\begin{vmatrix}1&1\\2&-1\end{vmatrix}-(-1)\cdot\begin{vmatrix}5&1\\3&-1\end{vmatrix}+2\cdot\begin{vmatrix}5&1\\3&2\end{vmatrix}=-6$$

Then,

$$x=\frac{\det(A_x)}{\det(A)}=\frac{-6}{-6}=1$$

Step 3: Compute $\det(A_y)$
Replace the second column of A by B

$$Ay=\begin{pmatrix}1&4&2\\2&5&1\\3&3&-1\end{pmatrix}$$

Then

$$\det(Ay) = 1 \cdot \begin{vmatrix} 5 & 1 \\ 3 & -1 \end{vmatrix} - 4 \cdot \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 5 \\ 3 & 3 \end{vmatrix} = -6$$

Then,

$$y = \frac{\det(A_y)}{\det(A)} = \frac{-6}{-6} = 1$$

Step 3: Compute $\det(A_z)$
Replace the third column of A by B

$$A_z = \begin{pmatrix} 1 & -1 & 4 \\ 2 & 1 & 5 \\ 3 & 2 & 3 \end{pmatrix}$$

Then

$$\det(A_z) = 1 \cdot \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 2 & 5 \\ 3 & 3 \end{vmatrix} + 4 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = -12$$

So,

$$z = \frac{\det(A_z)}{\det(A)} = \frac{-12}{-6} = 2$$

Therefore the solution is

$$(x, y, z) = (1, 1, 2)$$

Conclusion

This manuscript has presented a structured introduction to the essential concepts of linear algebra and mathematical tools required at the first-year university level. Throughout the five chapters, the main foundational notions have been progressively developed, starting from basic mathematical logic and set theory, and extending to vector spaces, linear applications, matrices, systems of linear equations, and polynomial rings.

The objective of this course support is not only to provide theoretical definitions and results, but also to help students develop a clear and rigorous mathematical reasoning. The progressive organization of the material allows the reader to move from fundamental abstract concepts to more applied and computational tools, illustrating the strong interaction between theory and practice in linear algebra.
Special attention has been given to the relationship between linear transformations and matrices, as well as to the study of linear systems, which represent a central application of the theory. These topics highlight the importance of algebraic structures in modeling and solving real mathematical problems.
In addition, the inclusion of exercises and examination models with detailed solutions and grading schemes aims to strengthen the student's autonomy and prepare them effectively for assessments. This pedagogical approach helps students not only to understand the content, but also to master the techniques required in examinations.

Finally, this manuscript should be viewed as a learning support rather than a reference book. It is intended to accompany students in their first steps in abstract algebra and to provide them with the necessary tools for further studies in mathematics and related fields. Mastering these concepts constitutes an essential foundation for more advanced topics in mathematics, physics, engineering, and other scientific disciplines.

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