EXO 1 :| 07

(E) : $u_{xx} + 6u_{xy} - 16u_{yy} = 0$.

1 - we show (E) is hyperbolic :|

we have : $a = 1$, $2b = 6$; $c = -16$.

$\Delta = b^2 - ac = (3)^2 - (1)(-16) = 9 + 16 = 25$ 0,25

Since $\Delta > 0$ so the equation (E) is hyperbolic 0,25

2 - The Canonic form of equation (E) :|

The characteristic equation :

$$\frac{dy}{dx} = \frac{b \pm \sqrt{\Delta}}{a} = \frac{3 \pm \sqrt{25}}{1}$$ 0,25

so $\begin{cases} \dfrac{dy}{dx} = 8 \\ \text{and } \dfrac{dy}{dx} = -2 \end{cases} \Leftrightarrow \begin{cases} y = 8x + c_1 \\ y = -2x + c_2 \end{cases}$

so $\begin{cases} \varphi_1(x,y) = 8x - y \\ \varphi_2(x,y) = 2x + y \end{cases}$ 0,25 0,25

so we put : $u(x,y) = v(s,t)$ 0,25 with : $\begin{cases} s = 8x - y \\ t = 2x + y \end{cases}$ 0,5

The Jacobien : 0,25 $J\begin{pmatrix} 8 & -1 \\ 2 & 1 \end{pmatrix} = 10 \neq 0$.

We have:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= v_s \cdot 8 + v_t \cdot 2$$

$$= \boxed{8v_s + 2v_t} \quad 0{,}28$$

$$u_{xx} = \frac{\partial}{\partial x}\left(8v_s + 2v_t\right)$$

$$= 8\frac{\partial}{\partial s}\left(8v_s + 2v_t\right) + 2\frac{\partial}{\partial t}\left(8v_s + 2v_t\right)$$

$$= 64 v_{ss} + 16 v_{st} + 16 v_{st} + 4 v_{tt}$$

$$\boxed{u_{xx} = 64 v_{ss} + \cancel{18}\ 32\, v_{st} + 4 v_{tt}} \quad 0{,}28$$

$$\frac{\partial u}{\partial y} = -v_s + v_t \quad 0{,}28$$

$$u_{yy} = \frac{\partial}{\partial y}\left(-v_s + v_t\right)$$

$$= \frac{\partial}{\partial s}\left(v_s - v_t\right) + \frac{\partial}{\partial t}\left(-v_s + v_t\right)$$

$$= v_{ss} - v_{ts} - v_{ts} + v_{tt}$$

$$\boxed{u_{yy} = v_{ss} - 2v_{ts} + v_{tt}} \quad 0{,}28$$

$$u_{xy} = \frac{\partial}{\partial x}\left(-v_s + v_t\right)$$

$$= \frac{\partial}{\partial s}\left(-8v_s - 2v_t\right) + \frac{\partial}{\partial t}\left(8v_s + 2v_t\right)$$

$= -8v_{ss} - \cancel{2}v_{st} + 8v_{st} + 2v_{tt}$

$= \boxed{-8v_{ss} + 6v_{st} + 2v_{tt}.}$ 0,2

So

$64v_{ss} + \cancel{18}\,32\,v_{st} + 4v_{tt} + 48v_{ss} + 36v_{st} + 12v_{tt}$

$-16v_{ss} + 32v_{st} - 16v_{tt}$

$= \cancel{86v_{st}}\ 100\,v_{st} = 0$ 0,1

$\iff v_{st} = 0.$

3| we find a general solution $u(x,y)$.

we have $v_{st} = 0 \iff v_s = g(s)$ 0,2

$\iff v(s,t) = G(s) + F(t)$ 0,2

$\iff u(x,y) = G(8x - y) + F(2x + y)$. 0,2

4| we Find a solution $u(x,y)$ satisfies:

$u(-x, 2x) = x$ and $u(x,0) = \sin(2x)$.

we have: $u(-x, 2x) = G(-10x) + F(0) = x$. 0,2

and: $u(x,0) = G(8x) + F(2x) = \sin(2x)$ 0,2

we have: $G(-10x) = x - F(0)$ $\cancel{x}x$.
0,2

so $F(0) = 0$ 0,25

~~when~~ so: $G(-10x) = x \Leftrightarrow \boxed{G(x) = \frac{-x}{10}}$

and we have: $G(8x) + F(2x) = \sin(2x)$. 0,25

$\Leftrightarrow F(2x) = \frac{4x}{5} + \sin 2x$ 0,25

$\Leftrightarrow \boxed{F(x) = \frac{2x}{5} + \sin x}$ 0,25

so: $u(x,y) = \frac{-8x+y}{10} + \frac{4x+2y}{5} + \sin(2x+y)$

$\Leftrightarrow \boxed{u(x,y) = \frac{y}{2} + \sin(2x+y)}$ 0,5

EXO2 :

$$\begin{cases} -y u_x + x u_y = u. \\ u(x,0) = \varphi(x). \end{cases}$$

1/ The characteristic system:

$$\begin{cases} \frac{dx}{dt} = -y. \\ \frac{dy}{dt} = x \\ \frac{du}{dt} = u. \end{cases}$$ 0,75

and: The initial condition

$$\begin{cases} x(0,s) = s \\ y(0,s) = 0. \\ u(0,s) = \varphi(s). \end{cases}$$ 0,75

2/ we show that:

$$\begin{cases} x(t,s) = s\cos t. \\ y(t,s) = s\sin t \\ u(t,s) = \varphi(s) \cdot e^{t}. \end{cases}$$

we have: $\frac{du}{dt} = u \Leftrightarrow \ln u = t + f_1(s)$

$\Leftrightarrow u = f_2(s) e^{t}$

$\Leftrightarrow \boxed{u(t,s) = \psi(s)\, e^{t}}$ 0,1.

we have: $\frac{dx}{dt} = -y \Leftrightarrow \frac{d^2x}{dt^2} = -\frac{dy}{dt}$ 0,25

$\Leftrightarrow \frac{d^2x}{dt^2} = -x$ 0,25 $\Leftrightarrow \frac{d^2x}{dt^2} + x = 0$ 0,25.

the caracteristic function is: $r^2 + 1 = 0$ 0,25

$\Leftrightarrow r = i$ or $r = -i$.

so: $x = f_1(s) \cos(t) + f_2(s) \sin(t)$ 0,5

so, $\boxed{x = s\cos(t) + f_2(s)\sin t}$ 0,5.

since: $\frac{dx}{dt} = -y \Leftrightarrow y = -\frac{dx}{dt}$ 0,5 $\Leftrightarrow \boxed{y = s \sin t}$ 0,5

$\Leftrightarrow y = s\sin t + f_2(s)\cos t$

3| we Find $u(x,y)$.

we have: $x = s\cos t$ and $y = s\sin t$.

$\Leftrightarrow s = \sqrt{x^2+y^2}$ 0,25 and: $\frac{y}{x} = \tan t$

so: $t = \arctan\left(\frac{y}{x}\right)$ 0,25

we have: $\boxed{u(x,y) = \psi\left(\sqrt{x^2+y^2}\right) e^{\arctan\left(\frac{y}{x}\right)}}$. 0,1

EX 03 : 1

1/ We write the compatibility Condition:

We have: $u(x,0) = f(x) = x(\pi - x)$.

0,75

$u(0,0) = f(0) = 0$ and $u(\pi,0) = f(\pi) = 0$. 0,25

So: $f(0) = f(\pi) = 0$.

2/

We put: $u(x,t) = X(x) \cdot T(t) \neq 0$. 0,25

The condition a limite:

$u(0,t) = X(0) \cdot T(t) = 0 \iff X(0) = 0$. 0,25

$u(\pi,t) = X(\pi) \cdot T(t) = 0 \iff X(\pi) = 0$. 0,25

We have: $u_{t} - u_{xx} = 0$

$\iff X(x) \cdot T'(t) - X''(x) \cdot T(t) = 0$

$\iff \dfrac{T'(t)}{T(t)} - \dfrac{X''(x)}{X(x)} = 0$ 0,25

$\iff \dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)}$

So $\exists \lambda \in \mathbb{R}$ such that:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$ 0,25

$\boxed{6}$

$\Leftrightarrow (P_1) \begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(\pi) = 0 \end{cases}$ 0,25 and $T'(t) = -\lambda T(t)$. 0,25

we solve (P_1).

if $\lambda = 0$ then $X(x) = \alpha + \beta(x)$. 0,25

we have $X(0) = \alpha = 0$.

and: $X(\pi) = \beta\pi = 0 \Leftrightarrow \beta = 0$.

so $X(x) \equiv 0$ (sol trivial).

if $\lambda < 0$ then $X(x) = \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x}$ 0,25

we have: $X(0) = 0 \Leftrightarrow \alpha + \beta = 0 \Leftrightarrow \alpha = -\beta$.

$$X(\pi) = 0 \Leftrightarrow \alpha\left(e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi}\right) = 0$$

$$\Leftrightarrow \alpha = 0$$

so $\alpha = \beta = 0$.

if $\lambda > 0$ then $X(x) = \alpha\cos(\sqrt{\lambda}x) + \beta\sin(\sqrt{\lambda}x)$

so: $X(0) = 0 \Leftrightarrow \alpha = 0$. 0,25

so $X(x) = \beta\sin(\sqrt{\lambda}x)$.

we have $X(\pi) = 0 \Leftrightarrow \beta\sin(\sqrt{\lambda}\pi) = 0$

$$\Leftrightarrow \sqrt{\lambda}\pi = n\pi$$ 0,25

$$\Leftrightarrow \sqrt{\lambda} = n \Leftrightarrow \boxed{\lambda = n^2}$$

so: $\boxed{X_n(x) = \beta_n \sin(nx)}$ 0,25

We have: $T_n(t) = C_n \, e^{-\alpha t} \Leftrightarrow \boxed{T_n(t) = C_n e^{-n^2 t}}$ 0,25

so; $u(x,t) = \sum_{n\geq 1} X_n(x) \cdot T_n(t)$

$= \sum_{n\geq 1} A_n \sin(nx) \, e^{-n^2 t}$ 0,25

C.I: $u(x,0) = \sum_{n\geq 1} A_n \sin(nx) = f(x)$ so;

where;

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin(nx)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin(nx) \, dx$$ 0,25

$x(\pi - x) \Rightarrow \pi - 2x$.

$\sin(nx) \longrightarrow \frac{-1}{n}\cos nx$.

$$= \left[\frac{2}{n\pi}\int_0^{\pi} (\pi - 2x)\cos(nx)\, dx \right.$$ 0,5

$\pi - 2x \longrightarrow -2$

$\cos nx \longrightarrow \frac{1}{n}\sin(nx)$

$$= \frac{4}{n^2\pi}\int_0^{\pi} \sin(nx)\, dx$$ 0,5

$$= \frac{-4}{n^3\pi}\left[\cos(nx)\right]_0^{\pi} = \frac{-4}{n^3\pi}\left[(-1)^n - 1\right]$$ 0,25 0,5

$$= \begin{cases} 0 & n = 2m \\ \dfrac{8}{(2m-1)^3\pi} & n = 2m-1 \end{cases}$$

0,5

So: $$u(x,t) = \sum_{m\geq 1} \frac{8}{(2m-1)^3\pi} \sin(2m-1)x \, e^{-(2m-1)^2 t}$$

0,5

$$\boxed{= \frac{8}{\pi} \sum_{m\geq 1} \frac{\sin[(2m-1)x]}{(2m-1)^3} e^{-(2m-1)^2 t}}$$