Université Larbi Ben M'hidi
Dept. SM
Maths3 - L2
Séries et Eqns. Diff.

Solution of Exam

13 janvier 2026 **Durée : 1h30**

Tous les documents, autres que ceux fournis dans le sujet, sont interdits. [1]

Solution 1 .

1. *Consider the ODE*

$$y' = (y-x)^2.$$

Isoclines. *An isocline is defined by the condition that the slope y' is constant. Here :*

$$(y-x)^2 = k \quad \Rightarrow \quad y - x = \pm\sqrt{k}.$$

So the isoclines are two straight lines parallel to the diagonal $y = x$:

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$$y = x + \sqrt{k}, \qquad y = x - \sqrt{k}.$$

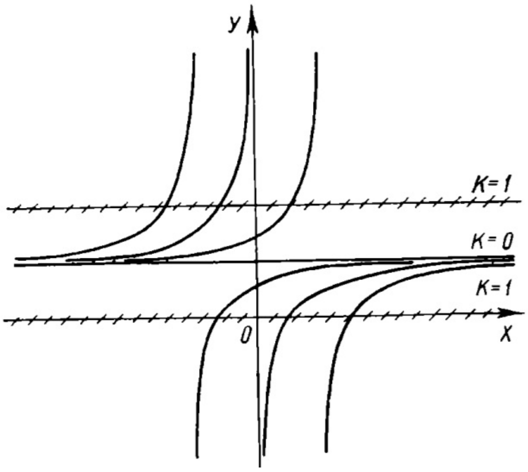

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Figure 1

Graphical interpretation. *- The isoclines $y = x \pm \sqrt{k}$ are straight lines where the slope is constant and equal to k. - Solutions can be drawn to show that they touch these isoclines : for example, the solution $y = x + 1$ is tangent to the isocline corresponding to $k = 1$. - In general, solution curves are increasing, horizontal when they pass through $y = x$, and become steeper as they move away from the diagonal.*

2. *For the ODE*

$$\frac{dy}{dx} = -\frac{2xy + y^2 + 1}{x^2 + 2xy},$$

write it as

$$(2xy + y^2 + 1)\,dx + (x^2 + 2xy)\,dy = 0.$$

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Let $M(x,y) = 2xy + y^2 + 1$, $N(x,y) = x^2 + 2xy$. Since

$$\frac{\partial M}{\partial y} = 2x + 2y, \qquad \frac{\partial N}{\partial x} = 2x + 2y,$$

the equation is exact.

1. Barème : 7-6-7 pts.

Integrating M with respect to x :

$$F(x,y) = x^2y + xy^2 + x + g(y).$$

Matching $\partial F/\partial y = N$ gives $g'(y) = 0$. Thus the implicit solution is

$$x^2y + xy^2 + x = C.$$

Equivalently, solving for y :

$$y(x) = \frac{-x^2 \pm \sqrt{x^4 - 4x(x - C)}}{2x}, \qquad x \neq 0.$$

Solution 2 .

1–Method of characteristics. *The PDE*

$$(y^2 - u^2)\, u_x + (u^2 - x^2)\, u_y = x^2 - y^2$$

admits the characteristic system

$$\frac{dx}{ds} = y^2 - u^2, \qquad \frac{dy}{ds} = u^2 - x^2, \qquad \frac{du}{ds} = x^2 - y^2.$$

Equivalently,

$$\frac{dx}{y^2 - u^2} = \frac{dy}{u^2 - x^2} = \frac{du}{x^2 - y^2}.$$ 1

Starting from

$$\frac{dx}{y^2 - u^2} = \frac{dy}{u^2 - x^2} = \frac{du}{x^2 - y^2},$$

summing numerators and denominators gives

$$\frac{dx + dy + du}{(y^2 - u^2) + (u^2 - x^2) + (x^2 - y^2)} = \frac{dx + dy + du}{0},$$

which again yields the first integral 1

$$U(x,y,u) = x + y + u.$$

Multiplying each fraction by x^2, y^2, and u^2 respectively, then summing, gives

$$\frac{x^2dx + y^2dy + u^2du}{0},$$

leading to the second integral 1

$$V(x,y,u) = x^3 + y^3 + u^3.$$

Independence of the integrals. *The gradients*

$$\nabla U = (1,1,1), \qquad \nabla V = (3x^2, 3y^2, 3u^2)$$

are linearly independent except on special loci. More formally, the Jacobian

$$J = \begin{pmatrix} 1 & 1 & 1 \\ 3x^2 & 3y^2 & 3u^2 \end{pmatrix}$$

has rank 2 for generic (x,y,u). Thus U and V are functionally independent.

General solution. *The invariants U, V remain constant along characteristics, so the general solution is given implicitly by*

$$F(U,V) = F(x + y + u,\ x^3 + y^3 + u^3) = 0,$$ 1

where F is an arbitrary smooth function.

*2–**Integral surface through the line** $(P): z=0,\ y=2x$. On (P) we have $u=0,\ y=2x$. Then*

$$U|_{(P)} = x + 2x + 0 = 3x = C_1, \qquad V|_{(P)} = x^3 + (2x)^3 + 0 = 9x^3 = C_2.$$ 1

Eliminating $x = C_1/3$ gives

$$C_2 = \tfrac{1}{3}C_1^3.$$

Hence the required integral surface is characterized by 1

$$V - \tfrac{1}{3}U^3 = 0, \quad i.e. \quad x^3 + y^3 + u^3 - \tfrac{1}{3}(x+y+u)^3 = 0.$$

Solution 3 .

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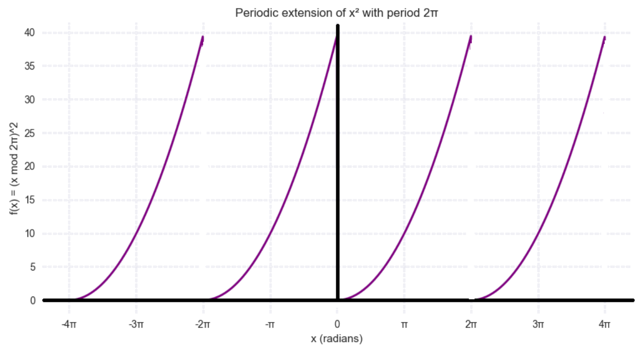

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FIGURE 2

2– We consider the 2π-periodic extension of $f(x) = x^2$ on $(0, 2\pi)$.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$ 0.5

Constant term :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{8\pi^2}{3}, \qquad \frac{a_0}{2} = \frac{4\pi^2}{3}.$$ 1

Cosine coefficients :

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos(nx)\,dx.$$ 0.5

Using integration by parts twice in one line :

$$\int_0^{2\pi} x^2 \cos(nx)\,dx = \left[\frac{x^2 \sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right]_0^{2\pi}.$$

Since $\sin(0) = \sin(2\pi n) = 0,\ \cos(2\pi n) = 1$, we obtain

$$\int_0^{2\pi} x^2 \cos(nx)\,dx = \frac{4\pi}{n^2}, \qquad a_n = \frac{4}{n^2}.$$ 1

Sine coefficients :

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin(nx)\,dx.$$ 0.5

Integration by parts twice :

$$\int_0^{2\pi} x^2 \sin(nx)\,dx = \left[-\frac{x^2 \cos(nx)}{n} + \frac{2x\sin(nx)}{n^2} + \frac{2\cos(nx)}{n^3}\right]_0^{2\pi}.$$

Boundary terms simplify to

$$\int_0^{2\pi} x^2 \sin(nx)\, dx = -\frac{4\pi^2}{n}, \quad b_n = -\frac{4\pi}{n}.$$ 1

Final Fourier series :

$$x^2 \sim \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos(nx) - \frac{4\pi}{n} \sin(nx) \right).$$ 0,5

3– ***Convergence at the endpoints :*** *The* 2π*-periodic extension has a jump at* $x = 0$ *(from* 0 *to* $4\pi^2$*), so by Dirichlet convergence the series equals the midpoint*

$$\frac{f(0^+) + f(0^-)}{2} = \frac{0 + 4\pi^2}{2} = 2\pi^2.$$

Setting $x = 0$ *in the series gives*

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$$2\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} \frac{4}{n^2} = 2\pi^2 - \frac{4\pi^2}{3} = \frac{2\pi^2}{3}.$$

Therefore,

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}.$$