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Final exam of the third semester in Alg 3
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Exorcise 1

1. Let E be a $\mathbb{R}-$vector space of dimension n and $f \in l(E)$ an endomorphism. Prove that the vector lines invariant under f are the $1-$dimensional vector spaces of the form $\ker(f-\lambda I)$ for some $\lambda \in \mathbb{R}$. $(1p)$

2. Deduce that λ is an eigenvalue of f. $(0,5p)$

3. Let
$$A = \begin{pmatrix} -1 & 2 & 1 & 0 \\ 2 & -1 & -1 & 0 \\ -4 & 4 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

   a) Determine all the vector lines invariant under A. $(4p)$
   b) Prove that A is diagonalizable. $(1,5p)$
   c) Deduce the normal Jordan form of A. $(0,5p)$
   d) Decompose $\mathbb{R}^4$ into direct sum of vector lines. $(0,5p)$

4. Solve the following differential system: $(2,5p)$
$$\begin{cases} x_1' &= -x_1 + 2x_2 + x_3 \\ x_2' &= 2x_1 - x_2 - x_3 \\ x_3' &= -4x_1 + 4x_2 + 3x_3 + x_4 \\ x_4' &= 2x_4 \end{cases}$$

5. Determine $A^{2026} + A^{-14} + e^A$. (not necessary to calculate P^{-1}). $(1,5p)$

Exorcise 2

Let B be a square matrix of order 4 such that $B^3 - 5B^2 + 8B - 4I = 0$ and $\dim\ker(B-2I) = 2$.

a) Determine all the possible minimal polynomials and characteristic polynomials of B. $(4p)$

b) Write the normal Jordan form of B in each case. $(4p)$

Solution

Exorcise 1

1. The vector lines invariant under f . Let D be a vector line generated by a vector $v \in D$. Then, $\forall\ X \in D$, $\exists \alpha \in \mathbb{R}$, $X = \alpha v$. that gives

$$f(X) = f(\alpha v) = \alpha f(v) \tag{1}$$

   Since D is invariant under f, we have $f(v) \in D$. Thus, there exists $\lambda \in \mathbb{R}$, such that $f(v) = \lambda v$. By replacing this last expression in (1), we get

$$f(X) = \alpha(\lambda v) = (\alpha\lambda)v = (\lambda\alpha)v = \lambda(\alpha v) = \lambda X \tag{2}$$

   From (2), we have

$$f(X) - \lambda X = 0 \Rightarrow (f - \lambda I)(X) = 0 \Rightarrow X \in \ker(f - \lambda I)$$

   Consequently,

$$D \subset \ker(f - \lambda I)$$

   Since $\dim\ker(f - \lambda I) = 1$, then

$$D = \ker(f - \lambda I).$$

2. From Question 1, we have

$$\ker(f - \lambda I) \neq 0 \Leftrightarrow \det(f - \lambda I). = 0 \Leftrightarrow \lambda \text{ is an eigenvalue of } f.$$

3. From Question 2, we have the following:

   a) The determination of the invariant vector lines of A is equivalent to the determination of all the eigenvalues $\lambda \in \mathbb{R}$, for which

   $\dim\ker(A - \lambda I). = 1$ or $\ker(A - \lambda I)$ is a direct sum of vector lines.

$$\begin{aligned}\det(A-\lambda I). &= 0 \Rightarrow \det\begin{pmatrix} -1-\lambda & 2 & 1 & 0 \\ 2 & -1-\lambda & -1 & 0 \\ -4 & 4 & 3-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix} = 0 \\ (2-\lambda)(\lambda+1)(\lambda-1)^2 &= 0 \Rightarrow \lambda_1 = 2,\ \lambda_2 = 1 = \lambda_3,\ \lambda_4 = -1.\end{aligned}$$

Since $\lambda_1 = 2$ and $\lambda_4 = -1$ are of multiplicity 1, then, the geometric multiplicities

$$\dim\ker(A-2I) = 1 \text{ and } \dim\ker(A+I) = 1$$

Therefore, we have at least two vector lines $D_1 = \dim\ker(A-2I)$ and $D_2 = \dim\ker(A+I)$. For the eigenvalue $\lambda_2 = 1$, we have to determine $\dim\ker(A-I)$.

$$\begin{pmatrix} -2 & 2 & 1 & 0 \\ 2 & -2 & -1 & 0 \\ -4 & 4 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \Rightarrow \left\{\begin{array}{rl} -2x+2y+z &= 0 \\ 2x-2y-z &= 0 \\ -4x+4y+2z+t &= 0 \\ t &= 0 \end{array}\right. \tag{3}$$

The solution of System (3) gives

$$(x,y,z,t) = (x,y,2x-2y,0) = x(1,0,2,0) + y(0,1,-2,0)$$

That gives

$$\ker(A-I) = vect\{(1,0,2,0)\} \oplus vect\{(0,1,-2,0)\},$$

Let $v = (1,0,2,0)$ and $D_3 = vect\{v\}$. Then, for every $x \in D_3$, there exists $\alpha \in \mathbb{R}$, such that $x = \alpha v$. Since $Av = v$, then $Ax = A\alpha v = \alpha Av = \alpha v \in D_3$, which means that D_3 is invariant under A. Now, let $u = (0,1,-2,0)$ and $D_4 = vect\{u\}$. By the same manner, D_4 is invariant under A. Therefore, we have 4 vector lines invariant under A.

b) From the results in a), we have 4 independent eigenvectors v_1, $v_2 = v$, $v_3 = u$, v_4 associated to the eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1 = \lambda_3$, $\lambda_4 = -1$ of the matrix A respectively, which constitute a basis for $\mathbb{R}^4$, then, A is diagonalizable.

c) The normal Jordan form of A: Since A is diagonalizable, then all Jordan blocks of A are of order 1, that gives:

$$P^{-1}AP = J_1(2) \oplus J_1(1) \oplus J_1(1) \oplus J_1(-1) = \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

d) Without care of the order,

$$\mathbb{R}^4 = D_1 \oplus D_2 \oplus D_3 \oplus D_4.$$

4. Since A is diagonalizable, then, the general solution of the given differential system is of the form

$$X(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3 t} v_3 + c_4 e^{\lambda_4 t} v_4 \tag{4}$$

where $v_1 \in \ker(A - 2I) \Rightarrow v_1 = (1, -1, 5, 3)$, $v_4 \in \ker(A + I) \Rightarrow v_4 = (1, -1, 2, 0)$, $v_2 = (1, 0, 2, 0)$, $v_3 = (0, 1, -2, 0) \in \ker(A - I)$.

By replacing the previous results in (4), we get

$$\begin{aligned} X(t) &= c_1 e^{2t}(1, -1, 5, 3) + c_2 e^{t}(1, 0, 2, 0) + c_3 e^{t}(0, 1, -2, 0) + c_4 e^{-t}(1, -1, 2, 0) \\ &= \left(c_2 e^{t} + c_1 e^{2t} + c_4 e^{-t}, c_3 e^{t} - c_1 e^{2t} - c_4 e^{-t}, 2c_2 e^{t} - 2c_3 e^{t} + 5c_1 e^{2t} + 2c_4 e^{-t}, 3 \right. \end{aligned}$$

5. Calculation of $A^{2026} + A^{-14} + e^{A}$:

$$\begin{aligned} A^{2026} + A^{-14} + e^{A} &= P\left(D^{2026} + D^{-14} + e^{D}\right) P^{-1} \\ &= P \begin{pmatrix} 2^{2026} + \frac{1}{2^{14}} + e^{2} & & & \\ & 2 + e & & \\ & & 2 + e & \\ & & & 2 + \frac{1}{e} \end{pmatrix} P^{-1}, \end{aligned}$$

where

$$P = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 5 & 2 & -2 & 2 \\ 3 & 0 & 0 & 0 \end{pmatrix}.$$

Exorcise 2

Since $B^3 - 5B^2 + 8B - 4I = 0$ and $\dim\ker(B - 2I) = 2$, then, $\lambda_1 = 2$ is an eigenvalue of B of algebraic multiplicity ≥ 2.

Let

$$P(X) = X^3 - 5X^2 + 8X - 4 = (X-2)^2(X-1)$$

a) i) The minimal polynomial of B:

Since $P(B) = B^3 - 5B^2 + 8B - 4I = 0$, then, the minimal polynomial $m_B(X)$ divides $P(X)$. Thus, we have two possibilities for $m_B(X)$, that are:

$$m_B(X) = (X-2)^2(X-1) \text{ or } m_B(X) = (X-2)(X-1)$$

ii) Since B is of order 4 and $P(X)$ of order 3 satisfies $P(B) = 0$, then, the characteristic polynomial $C_B(X)$ is of degree 4 and $P(X)$ divides $C_B(X)$. Thus, we get

$$C_B(X) = P(X)(X-1) = (X-2)^2(X-1)^2 \text{ or } P(X)(X-2) = (X-2)^3(X-1)$$

b) Normal Jordan form of B:

- If $C_B(X) = (X-2)^3(X-1)$ and $m_B(X) = (X-2)(X-1)$, then, the matrix is diagonalizable and thus, $\dim\ker(B - 2I) = 3$, which to be rejected.
- If $C_B(X) = (X-2)^3(X-1)$, then, $m_B(X) = (X-2)^2(X-1)$, then, the normal Jordan for of B is

$$Q^{-1}BQ = J_2(2) \oplus J_1(2) \oplus J_1(1) = \begin{pmatrix} 2 & 1 & & \\ 0 & 2 & & \\ & & 2 & \\ & & & 1 \end{pmatrix} \quad (5)$$

- If $C_B(X) = (X-2)^2(X-1)^2$ and $m_B(X) = (X-2)(X-1)$, then, the normal Jordan form is:

$$Q^{-1}BQ = J_1(2) \oplus J_1(2) \oplus J_1(1) \oplus J_1(1) = \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad (6)$$

- If $C_B(X) = (X-2)^2 (X-1)^2$ and $m_B(X) = (X-2)^2 (X-1)$, then, the normal Jordan form is:

$$Q^{-1}BQ = J_2(2) \oplus J_1(1) \oplus J_1(1) = \begin{pmatrix} 2 & 1 & & \\ 0 & 2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

which yields to $\dim\ker(B-2I) = 1$, which to be rejected.

Conclusin: From (5) and (6), we have two possible normal Jordan forms for B, with

$$C_B(X) = (X-2)^3 (X-1), \; m_B(X) = (X-2)^2 (X-1), \; \dim\ker(B-2I) = 2$$

or

$$C_B(X) = (X-2)^2 (X-1)^2, \; m_B(X) = (X-2)(X-1), \; \dim\ker(B-2I) = 2$$