L'arbi Ben M'hidi Oum El Bouaghi university
Department of mathematics **Teacher : SOULA**
University year 2025-2026 **Module: Algebra 01**
Time: 1h 30 m

Exam in: 17/01/2026

Exercise 1. (6 points)
Prove that by recurrence
a) $2^n > n^{\,2}$ for all $n \geq 5$.
b) $n! > 2n$ for all $n \geq 4$.

Exercise 2 (6 points)
Let $\Re$ be a relation on $\mathbb{Z}$ defined by:

$$x \Re y \Leftrightarrow x + y \quad \text{is an even number.}$$

a) Determine whether $\Re$ is reflexive.
b) Determine whether $\Re$ is symmetric.
c) Determine whether $\Re$ is transitive.
d) Deduce the type of the relation $\Re$
e) Determine the equivalence class 0, 1.
.
Exercise 3. (8 points)
Let

$$R = \{a + b\sqrt{2} \mid a, b \in Z\}$$

with the usual addition and multiplication.
a) Prove that $(R, +, \times)$ is a ring.
b) Is $(R, +, \times)$ commutative?
c) Does $(R, +, \times)$ have a unity element?

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The ideal solution for the exam

Exercise 1. (6 points)
Proof by recurrence
a) $2^n > n^2$ for all $n \geq 5$.
For $n = 5$, $2^5 = 32$, $5^2 = 25$, and $32 > 25$,..................(0.5 pt)
so the base case holds.
Inductive step: Assume for some.....................................(0.5 pt)

$$n \geq 5,\ 2^n > n^2.$$

We want to prove that:

$$2^{n+1} > (n+1)^2$$

Start from the left:
$2^{n+1} = 2 \cdot 2^n > 2 \cdot n^2$ (by inductive hypothesis).
It suffices to show that
$2n^2 \geq (n+1)^2 = n^2 + 2n + 1$,
or equivalently,
$2n^2 - n^2 - 2n - 1 = n^2 - 2n - 1 \geq 0$.
Check this inequality for
$n \geq 5$:
$n^2 - 2n - 1 = (n-1)^2 - 2 \geq 16 - 2 = 14 > 0$,
so it holds.
Therefore,
$2^{n+1} > (n+1)^2$.
By induction, the inequality holds for all $n \geq 5$.....................(2 pts)

b) $n! > 2n$ for all $n \geq 4$.
Base case: For $n = 4$,
$4! = 24$, $2(4) = 8$, $24 > 8$, ...(0.5 pt)
So the base case holds.
Inductive step: Assume for some $n \geq 4$,.....................................(0.5 pt)

$$n! > 2n$$

We want to prove that

$$(n+1)! > 2(n+1).$$

Starting from the left:

$$(n+1)! = (n+1)n! > 2n\cdot(n+1) > 2(n+1).$$

So,

$$(n+1)! > 2(n+1)^{\cdot}$$

By induction, the inequality holds for all $n \geq 4$..........................(2 pt)

<u>**Exercise 2**</u> (6 points)

a) Reflexivity

For any $x \in Z$

$x + x = 2x$

which is even. Hence, $x\Re x$ for all x

So, $\Re$ is reflexive...(1 pt)

b) Symmetry

If $x\Re y$, then $x + y$ is even. Since

$y + x = x + y$,

it follows that $y\Re x$

Therefore,

$\Re$ is symmetric.............. ....(**1** pt)

c) Transitivity

Assume

$x\Re y$ and $y\Re z$. Then:

$x + y$ is even, and $y + z$ is even.

Adding these two equalities:

$(x + y) + (y + z) = x + z + 2y$ which is even. Hence,

$x + z$ is even, and therefore $x\Re z$

Thus,

$\Re$ is transitive...(**1** pt)

d) Conclusion

Since $\Re$ is reflexive, symmetric, and transitive,

it is an equivalence relation.................................(**1** pt)

e)

Equivalence class of 0

$$\overline{0} = \{x \in Z \mid x + 0 \text{ is even}\}$$

So,

$\overline{0} = \{\text{all even integers}\}$..............................(**1** pt)

Equivalence class of 1

$$\overline{1} = \{x \in Z \mid x + 1 \text{ is even}\} = \{x \in Z \mid x \text{ is odd}\}.$$

So,

$\overline{1} = \{\text{all odd integers}\}$...............................(**1** pt)

Exercise 3.

a) Ring axioms

Closed under addition and multiplication.............(**1 pt**)

Addition is **associative** and **commutative**...........(**1 pt**)

Additive identity:

$0 = 0 + 0\sqrt{2}$..(**1 pt**)

Additive inverse:

$-(a + b\sqrt{2}) = -a \ -b\sqrt{2}$...............................(**1 pt**)

Multiplication is associative.......................(**1** pt)

Distributive laws hold...(1 pt)

Hence,

R is a ring.

b) Commutativity...(1 pt)

Multiplication is commutative since

$(a + b\sqrt{2})(c + d\sqrt{2}) = (c + d\sqrt{2})(a + b\sqrt{2})$.

c) Unity

The unity element is..(**1 pt**)

$$1 = 1 + 0\sqrt{2}.$$