University of Oum El Boughi

Faculty of ES and NLS | Departement of M-CS

L3 Mathematics | **Module NVS**

Exam, January 2026

Exercise 1(4 pts). Let $(E; \|\cdot\|_E)$ and $(F; \|\cdot\|_F)$ be two Banach spaces. Let $T : E \to F$ a linear operator satisfies (1)

$$\forall x_n \in E : \text{if} x_n \to x \, \text{and} \, Tx_n \to y \Rightarrow y = Tx. \tag{1}$$

1. Prove that $(E; \|.\|_T) : \|x\|_T = \|x\|_E + \|Tx\|_F$ is a Banach space and deduce that T is continuous.
2. Let consider $E = C^1([0,1]); \mathbb{R})$ and $F = C^0([0,1]); \mathbb{R})$ with the norm $\|\cdot\|_\infty$. Prove that T satisfies (1) but T is not continuous.$(f_n(x) = x^n)$
3. Why do we get two different results ?

Exercice 2(9 pts). Let be $(E, \|\cdot\|)$ a $\mathbb{K}$ -normed space , $0 \neq \phi : E \to \mathbb{K}$ a linear form and $H = ker\phi$.
1. Prove that if ϕ is continuous we get :
a)

$$\frac{|\phi(x)|}{\|\phi\|_{E^*}} \leq d(x, H).$$

b). Forall $(b, x) \in E \backslash H \times E$, (remarking that $x = \frac{\phi(x)}{\phi(b)} b + x - \frac{\phi(x)}{\phi(b)} b$,) there exist $(x_n) \subset E \setminus H$, $(t_n) \subset \mathbb{R}$ and $(h_n) \subset H$ such that $\|x_n\| = 1$ and $x = h_n + t_n x_n$. Prove that for all $x \in E$, we have

$$\frac{|\phi(x)|}{\|\phi\|_{E*}} = d(x, H).$$

2. Let be $E = \ell_0$ be the vector space of real sequences having a finite number of non-zero terms $x = (x_n)_{n \in \mathbb{N}}$, and $\phi(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} x_i$. Show in which case we have : for all $x \in E \setminus H$ there exists $h \in H : \|x - h\| = d(x, H)$.
a). If E is equipped by the norm $\|x\|_\infty$.
b). If E is equipped by the norm $\|x\|_2 = (\sum_{i=0}^{\infty} |x_i|^2)^{1/2}$.

Exercice 3(9 pts). Let be $H = (L^2([0,1]); \mathbb{R})$ is NVS with the norm $\|\cdot\|_2$.
1. Prove that H is an Hilbert space.
2. Let be $F = \mathbb{R}_2[X]$ the space of real polynomials on $[0,1]$ of degree ≤ 2. Let P_F be the orthogonal projection of H onto F. Show that
$\forall f \in H, inf_{a_i}\{\int_0^1 |f(x) - \sum_{i=0}^2 a_i x^i|^2; a_i \in \mathbb{R}\}$ is attained at $y = P_F(f)$.
3. We now assume that $H = L^2(\mathbb{N}; \mathbb{R}) := (\ell^2; \mathbb{R})$. For n is a fixed integer,
3.1. We define ϕ by

$$\forall x \in H, \phi(x) = \sum_{i=0}^{n} \frac{x_i}{2^i},$$

Show that ϕ is continuous and calculate $\|\phi\|_{H^*}$ in two different ways.
3.2. Let be

$$M = \{x \in H; \sum_{i=0}^{n} \frac{x_i}{2^i} = 0.\}$$

Show that M has a closed complement sub space to be determined.
3.3. Give the distance from the element $(1, 0, 1, \cdots, 0)$ to M.

Note : The grade for question 1 of exercise 1 is considered the grade for the individual work.

L3 math. 95

Correction of NVS Exam (2025/2026)

$\frac{20}{20}$

Ex 1:

1°) a) Let be (x_n) is a Cauchy sequence in $(E, \|\cdot\|_T)$, then $\forall \varepsilon > 0, \exists n_0, \forall n \geq n_0 \Rightarrow \|x_n - x_m\|_T < \varepsilon$

Then $\|x_n - x_m\|_E + \|Tx_n - Tx_m\|_F < \varepsilon \Rightarrow \|x_n - x_m\| < \varepsilon$ and $\|Tx_n - Tx_m\| < \varepsilon$, So

(1) $\exists (x, y) \in E \times F$: $x_n \to x$ and $Tx_n \to y$ (Because E, F are Banach)

or from ① we get $y = Tx$. Then $\|x_n - x\|_T \to 0 \Rightarrow (E, \|\cdot\|_T)$ is a Banach.

b) From 1°. a) we get E, F are Banach $\Rightarrow (E \times T(E))$ is a Banach $\|\cdot\|_T$

So if $x_n \xrightarrow{(E, \|\cdot\|_E)} x \Rightarrow x_n \xrightarrow{\|\cdot\|_T} x$.

(1) Let be $y_n = \frac{x}{n\|x\|_E} \xrightarrow{\|\cdot\|_E} 0 \Rightarrow y_n$ is convergente in $(E, \|\cdot\|_T)$; it is a bounded sequence, then $\forall n \geq 1 \Rightarrow \exists C > 0$: $\frac{\|x\|_T}{n\|x\|_E} \leq C$ (for $n = 1$) $\Rightarrow \frac{\|x\|_T}{\|x\|_E} \leq C$

then $\|\cdot\|_T \sim \|\cdot\|_E$. So $\|Tx\| \leq (C-1)\|x\|_E$ i.e. T is continuous.

(1) 2°. $\exists T: (C^1([0,1]), \mathbb{R}) \to (C([0,1]); \mathbb{R}): f \mapsto Tf = f'$. where T satisfies ①

if $f_n \xrightarrow{\|\cdot\|_\infty} f$ and $f'_n \xrightarrow{\|\cdot\|_\infty} g$ So (From Weierstrass Theorem) $g = f'$.

but T is not continuous because $\exists f_n(x) = x^n \xrightarrow{\|\cdot\|_\infty} 0$ but $f'_n(x) \xrightarrow{\|\cdot\|_\infty} +\infty$

which means T is not continuous.

(1) 3°. We get two different result because E, and F in this case not Banach.

Ex 2: (9)

(3) 1°. a) φ is continuous then $\|\phi\|_{E^*} < +\infty$ and $\forall x \in E$, $|\phi(x)| \leq \|\phi\|_{E^*} \cdot \|x\|$.

Then $|\phi(x-h)| \leq \|\phi\|_{E^*} \cdot \|x-h\|$ $\forall (x,h) \in E \times \operatorname{Ker}\phi \Rightarrow \frac{|\phi(x)|}{\|\phi\|_{E^*}} \leq \inf \|x-h\| = d(x, H)$. ① 0,5

b) We have $\|\phi\|_{E^*} := \sup_{\|x_n\| = 1} |\phi(x_n)|$.

Then $\forall \frac{1}{n} > 0$ $\exists (b_n) \in E$: $\|x_n\| = 1$ and $\|\phi\|_{E^*} - \frac{1}{n} < |\phi(b_n)|$ ...... ② 0,5

In other hande $\forall x \in E$, $x = \frac{\phi(x)}{\phi(b_n)} b_n + x - \frac{\phi(x)}{\phi(b_n)} b_n$ & $\varphi(x - \frac{\varphi(x)}{\varphi(b_n)} b_n) = 0$

then we consider $h_n = \frac{\varphi(x)}{\varphi(b_n)} b_n$ and $b_n = x_n$ and $t_n = \frac{\phi(x)}{\varphi(b_n)}$ So:

$\forall x \in E$ $x = t_n x_n + h_n$; $\|x_n\| = 1$ ...... ② 0,5

• From ② we have $\|x - h_n\| = |t_n| = \frac{|\phi(x)|}{|\phi(b_n)|} \overset{②}{<} \frac{|\phi(x)|}{\|\phi\|_{E^*} - \frac{1}{n}}$ 0,5

Because $x \mapsto \|x\|$ is continuous and $\operatorname{Ker} T = H$ is closed (ϕ is continuous)

then by letting $n \to +\infty$ we get: $\inf \|x - h_n\| := d(x, H) \leq \frac{|\phi(x)|}{\|\phi\|_{E^*}}$ ...... ③ 0,5

From ① and ③ we get $d(x, H) := \frac{|\phi(x)|}{\|\phi\|_{E^*}}$. 0,5

3 2°/a) E is equipped by $\|\cdot\|_\infty$.

First we have $|\phi(x)| \leq \|x\|_\infty \left(\sum_{i=0}^{\infty} \frac{1}{2^i}\right) := \|x\|_\infty 2\left(1 - \left(\frac{1}{2}\right)^{N+1}\right)$. $\Rightarrow \|\phi\|_* \leq 2\left(1 - \frac{1}{2^{N+1}}\right)$ ... 0,5

Then exist $x_0 = (1, \ldots, \underset{\substack{\uparrow \\ N}}{1}, 0, 0, \ldots, 0)$ such that $\|x_0\|_\infty = 1$. 0,5

then $\phi(x_0) = \sum_{i=0}^{N} \frac{1}{2^i} = 2\left(1 - \frac{1}{2^{N+1}}\right) \Rightarrow \|\phi\|_* = 2\left(1 - \frac{1}{2^{N+1}}\right)$ ... 0,5

Then if we suppose that $\exists h \in H : \|x - h\| = d(x, H) \overset{①}{=} \frac{|\phi(x)|}{\|\phi\|_{E^*}} = \frac{|\phi(x)|}{2} = \max|x_i - h_i|$ 0,5

Which means that $2|x_i - h_i| \leq \frac{1}{2^i}|x_i - h_i|$ so $x_i = h_i$ contradiction with 0,5

$x = (x_i)_i \in E \setminus H$. Then if E is equipped by $\|\cdot\|_\infty$ $\nexists h \in H : d(x, H) = \|x - h\|$. 0,5

3 b) If E is equipped by $\|\cdot\|_2$ we have $|\phi(x)| \leq \left(\sum_{i=0}^{\infty} |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=0}^{\infty} \frac{1}{2^{2i}}\right)^{\frac{1}{2}} \leq C\|x\|_2$ 0,5

where $C = \left(\sum_{i=0}^{\infty} \frac{1}{2^{2i}}\right)^{\frac{1}{2}} \leq \left(4 \frac{1}{3}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{3}}$.

Then $H = \operatorname{Ker}\phi$ is closed and convex (Subspace). 0,5

So: $\forall x \in E \setminus H \Rightarrow \exists P_H(x) \in H : x - P_H(x) \perp H$ ... by Projection Theorem 0,5

In one hand $d(x, H) = \inf_{h \in H} \|x - h\| \leq \|x - P_H(x)\|$ ... ① 0,5

in other hand $\|x - h\|^2 = \|x - P_H(x)\|^2 + \|P_H(x) - h\|^2$ (Pythagore's) ... ② 0,5

Then $\forall x \in E \setminus H \Rightarrow \exists h \in H : \|x - h\| := \|x - P_H(x)\| = d(x, H)$ ... 0,5

EX3: 9

① 1/ $(H, \|\cdot\|_2)$ is a Hilbert space If $\|\cdot\|_2$ satisfies the parallelogram equality i.e.

$\forall (x, y) \in H^2 : \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

as $\forall (x, y) \in L^2$ we have: $\left(\|x + y\|^2 = \int_0^1 |x + y|^2 + \|x - y\|^2 = \int_0^1 |x - y|^2\right) = 2(\|x\|^2 + \|y\|^2)$

So H is a Hilbert space.

② 2/ $F = \mathbb{R}_2[X]$ is a subspace of dimension $= 3$. Then it is closed and convex. 0,5

Thanks to Projection Theorem we have: 0,5

$\forall f \in H, \exists! P_F(y) \in F : d(x, F) := \inf_{g \in F} \int_0^1 |f - g|^2$; $g(x) = \sum_{i=0}^{2} a_i x^i$ 0,5

Then $d(x, F)^2 = \inf \|y - y\|_2^2 := \inf_{a_i} \int_0^1 (f(x) - \sum a_i x^i)^2 \xrightarrow[2^\circ b \to Ex2]{} = \|y - P_F(y)\|^2$. 0,5

③ 3°-1: $|\phi(x)| \leq \left(\sum_{i=0}^{n} |x_i|^2\right)^{\frac{1}{2}} \left(\underbrace{\sum_{i=0}^{n} \frac{1}{2^{2i}}}_{C_0}\right)^{\frac{1}{2}} \leq \frac{4}{3}\|x\|_2$. Then ϕ is continuous ... ① 0,25

1,5 → First way, From ① we $\|\phi\|_{E^*} \leq \left(\sum_{i=0}^{n} \frac{1}{2^{2i}}\right)^{\frac{1}{2}} = C_0 = \left(\frac{4}{3}\left(1 - \left(\frac{1}{4}\right)^{n+1}\right)\right)^{1/2}$: geometric series 0,5

let $x_0 = \frac{1}{C_0}\left(1, \frac{1}{2}, \dots, \frac{1}{2^i}, \dots, \frac{1}{2^n}, 0, \dots, 0\right)$ $\|x_0\|_2 = 1$ and 0,5

$\phi(x_0) = \frac{1}{C_0} \sum_{i=0}^{n} \frac{1}{2^{2i}} = \frac{C_0^2}{C_0} = C_0 \Rightarrow \|\phi\|_{E^*} \geq C_0$. Then $\|\phi\|_{E^*} = C_0$ 0,5

1,5 → Second way. Because $(l^2, \mathbb{R})$ is a Hilbert space. Thanks to Riesz representation Theorem we have: $\forall x \in l^2, \forall \phi \in (l^2)^* \Rightarrow \exists y_\phi \in l^2 : \phi(x) = \langle x, y_\phi \rangle$: $\|\phi\|_{E^*} = \|y_\phi\|$.

By identification we get $y_\phi = \left(1, \frac{1}{2}, \dots, \frac{1}{2^i}, \dots, \frac{1}{2^n}, 0, \dots\right)$ 0,5

such that $\|y_\phi\| = \left(\sum_{i=0}^{n} \frac{1}{2^{2i}}\right)^{\frac{1}{2}} = \left(\frac{4}{3}\left(1 - \left(\frac{1}{4}\right)^{n+1}\right)\right)^{\frac{1}{2}}$. 0,5

1,5 3°-2 we remark that $M = \phi^{-1}(0)$ is a closed subspace of l^2. So it has closed complement subspace $M^\perp$: $\dim M^\perp = 1$, as $\phi(y_\phi) \neq 0$ so $M^\perp = [y_\phi]$ ($\forall x \in l^2$, $x = P_M(x) + \alpha y_\phi$) 0,5

1,5 3°-3. Because $l^2 = M \oplus [y_\phi] \Rightarrow \forall x \in l^2 \Rightarrow x = P_M(x) + \alpha y_\phi$; $y_\phi \perp P_M(x) \Rightarrow \langle x, y_\phi \rangle = \alpha\|y_\phi\|^2$ 0,25 0,25 0,25 0,5

let $x_0 = (1, 0, 1, \dots)$ then $\langle x_0, y_\phi \rangle = \left(1 + \frac{1}{4}\right) = \alpha\|y_\phi\|^2 \Rightarrow \alpha = \frac{5}{4\|y_\phi\|^2}$ 0,5

From 2° we have $d(x, M) = \|x_0 - P_M(x_0)\| = |\alpha|\,\|y_\phi\| := \frac{5}{4}\|y_\phi\|^{-1} = \frac{5}{4}\left(\sum_{i=0}^{n} \frac{1}{2^{2i}}\right)^{-\frac{1}{2}}$. 0,5