Larbi Ben Mhidi University Oum El Bouaghi.
Faculty of exact sciences,natural and life sciences.

Level : L1 Mathematic.
Duration : $\lim_{x\longrightarrow 0} \dfrac{3\sin x}{2x}$.
Academic year 2025-2026.

Exam Analysis1

Exercise 1 (5*pts*) *The five questions are independentent:*
Prove that

1. $\inf A = 1$, *where* A *is a set defined by* $= \left\{ \frac{n+1}{n-1}; n \geq 2 \right\}$.
2. $\forall x \in \mathbb{R}; \forall n \in \mathbb{Z}; E(x+n) = E(x) + n$.
3. *If a function* f *accepts a limit* l *at* x_0*; then this limit is unique.*
4. $\forall x > 0; \dfrac{1}{x+1} < \ln(x+1) - \ln(x) < \dfrac{1}{x}$.
5. $\arctan(x) + \arctan\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$.

Exercise 2 (7.5*pts*) *Let* $(u_n)_{n\in\mathbb{N}}$ *be a real sequence defined by*

$$\begin{aligned} u_0 &= 2 \\ u_{n+1} &= \frac{1+u_n^2}{2u_n}. \end{aligned}$$

1. *Prove that* $\forall n \in \mathbb{N}; u_n \geq 1$.
2. $\left\{ \begin{array}{c} a) \text{ Study the monotonicity of the sequence } (u_n)_{n\in\mathbb{N}} \\ b) \text{ Deduce that : } \forall n \in \mathbb{N};\ u_n \leq 2. \end{array} \right.$.
3. *Show that :* $\forall n \in \mathbb{N}; u_{n+1} - 1 \leq \dfrac{1}{2}(u_n - 1)$.
4. *Prove that :* $\forall n \in \mathbb{N}; u_n - 1 \leq \left(\dfrac{1}{2}\right)^n$ *and deduce* $\lim_{n\longrightarrow\infty} u_n$.
5. *Let* (v_n) *be a real sequence defined by*

$$\forall n \in \mathbb{N};\ v_n = \frac{1}{u_n}.$$

Prove that (u_n) *and* (v_n) *are adjacent.* (*Turn the page please*).

Exercise 3 (7.5*pts*) *I*) *Using Hôpital rules, calculate* $\lim_{x\longrightarrow 0} \dfrac{e^x - 1}{x\,(e^x+1)}$.

II) *Let be* f *a real function defined on* $\mathbb{R}$ *by*

$$f(x) = \begin{cases} \dfrac{e^x - 1}{(e^x+1)}, & \textit{if} \quad x \leq 0 \\ ax & \textit{if} \quad x > 0 \end{cases} \qquad , a \in \mathbb{R}$$

1. *Study the continuity of* f *at* $x_0 = 0$.
2. *Find the value of* a *such that* f *is differentiable at* $x_0 = 0$, *and express* $f"(x)$ *in terms of* x.
3. *Show that* f *is bijective to* $\mathbb{R}$ *in the interval* $f(\mathbb{R})$ *wich must be determined.*
4. *Express* $f^{-1}(x)$ *in terms of* x.
5. *Calculate* $\left(f^{-1}\right)'(1)$.

Good luck.

Exo 1 (5 pts).

1 $A = \left\{ \frac{n+1}{n-1}, n \geq 2 \right\}$

0,25 we have: $\frac{n+1}{n-1} = 1 + \frac{2}{n-1} > 1$, $\forall n \geq 2$.

On the other side:

$\varepsilon > 0$; $\frac{n+1}{n-1} = 1 + \frac{2}{n-1} < 1 + \varepsilon \iff \frac{2}{n-1} < \varepsilon$

0,5 $\iff n-1 > \frac{2}{\varepsilon}$

$\iff n > \frac{2}{\varepsilon} + 1$.

0,25 and it's satisfied according to Archimed's Axiom.

So: $\inf A = 1$.

$\boxed{2}$ Prove that: $\forall x \in \mathbb{R}$, $\forall n \in \mathbb{Z}$, $E(x+n) = E(x)+n$

0,25 we have: $E(x) \leq x < E(x)+1$.

$\Updownarrow$

0,5 $\underbrace{E(x)+n}_{k \in \mathbb{Z}} \leq x+n < \underbrace{E(x)+n+1}_{k+1 \in \mathbb{Z}}$.

0,25 So: $E(x+n) = k = E(x)+n$.

[other methods are also acceptable].

$\boxed{3}$ [$\lim f(x)$]

we suppose that f accepts two limits l and l', so: $(l > l')$:

0,25 $\forall \varepsilon > 0$; $\exists \delta_1 > 0$; $\forall x \in V_{x_0}$; $|x - x_0| < \delta_1 \Rightarrow |f(x) - l| < \varepsilon$

0,25 $\forall \varepsilon > 0$; $\exists \delta_2 > 0$; $\forall x \in V_{x_0}$; $|x - x_0| < \delta_2 \Rightarrow |f(x) - l'| < \varepsilon$.

we choose $\varepsilon = \frac{l - l'}{2}$, and we put $\delta = \max(\delta_1, \delta_2)$.

0,25 So: $l - l' = |l - l'| = |f(x) - l' - f(x) + l|$

$\leq |f(x) - l| + |f(x) - l'|$

$< \frac{l - l'}{2} + \frac{l - l'}{2}$

0,25 Then $|l - l'| < l - l'$ (Contradiction).

So: $l = l'$.

1

4/ we put: $f(x) = \ln x$ and we choose

$I =]x ; x+1[$

- f is continuous at $[x, x+1]$

0,25 - f is differentiable at $]x ; x+1[$.

- Applaying the mean value theorem:

$\exists c \in]x ; x+1[\;;\; f'(c) = \dfrac{f(x+1) - f(x)}{x+1-x}$.

0,5 $f'(x) = \dfrac{1}{x} \;/\; \Rightarrow f'(c) = \dfrac{1}{c}$.

So: $\dfrac{\ln(x+1) - \ln x}{1} = \dfrac{1}{c}$

0,25 $x < c < x+1 \Rightarrow \dfrac{1}{x+1} < \dfrac{1}{c} < \dfrac{1}{x}$.

Then: $\dfrac{1}{x+1} < \ln(x+1) - \ln x < \dfrac{1}{x}$.

⑤ $\forall x > 0$ $\quad \arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}$.

0,25 we put: $f(x) = \arctan x + \arctan \dfrac{1}{x}$.

0,25 $f'(x) = \dfrac{1}{1+x^2} + \dfrac{-1/x^2}{1+\left(\frac{1}{x}\right)^2} = \dfrac{1}{1+x^2} - \dfrac{1}{1+x^2} = 0$

0,25 Then f is a constant function: $f(x) = f(1)$.

0,25 $= \arctan 1 + \arctan 1$

$\pi \quad \pi \quad \pi$

2

Exo 2:

$$\begin{cases} u_0 = 2 \\ u_{n+1} = \dfrac{1+u_n^2}{2u_n} \end{cases}$$

0,25 1/ $u_0 = 2 \geq 1$, $P(0)$ is verified.

0,25 Assume that $u_n \geq 1$; and prove that $u_{n+1} \geq 1$.

0,25x4 $u_{n+1} - 1 = \dfrac{1+u_n^2}{2u_n} - 1 = \dfrac{u_n^2 - 2u_n + 1}{2u_n} = \dfrac{(u_n-1)^2}{2u_n} \geq 0$

So: $u_{n+1} - 1 \geq 0$; then $u_{n+1} \geq 1$.

$\forall n \in \mathbb{N}$; $u_n \geq 1$.

2

0,25x4 a) $u_{n+1} - u_n = \dfrac{1+u_n^2}{2u_n} - u_n = \dfrac{1-u_n^2}{2u_n} \leq 0$ $\Big(u_n \geq 1;$

$1 - u_n^2 = (1-u_n)(1+u_n) \leq 0 \Big)$.

So (u_n) is decreasing.

b) (u_n) is decreasing; so: $\forall n \geq 0$; $u_n \leq u_0$. 0,25

0,5 Then: $u_n \leq 2$; $\forall n \in \mathbb{N}$.

0,25

3

3) 0,25x2 $u_{n+1} - 1 = \dfrac{(u_n-1)^2}{2u_n} \leq \dfrac{(u_n-1)^2}{2}$ $\left(u_n \geq 1 \Rightarrow \dfrac{1}{u_n} \leq 1 \right)$

0,25x2 $\leq \dfrac{u_n - 1}{2}$ $\left| \begin{array}{l} 1 \leq u_n \leq 2 \Rightarrow \\ 0 \leq u_n - 1 \leq 1 \Rightarrow \\ (u_n-1)^2 \leq (u_n-1) \end{array} \right.$

So

$$u_{n+1} - 1 \leq \frac{1}{2}(u_n - 1).$$

4

0,25 $u_0 - 1 = 2 - 1 \leq \left(\frac{1}{2}\right)^0 = 1$.

0,25 Assume that $u_n - 1 \leq \left(\frac{1}{2}\right)^n$.

0,5x2 $u_{n+1} - 1 \leq \frac{1}{2}(u_n - 1) \leq \frac{1}{2}\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{n+1}$

So: $\forall n \in \mathbb{N}$; $u_n - 1 \leq \left(\frac{1}{2}\right)^n$.

$0 \leq u_n - 1 \leq \left(\frac{1}{2}\right)^n$.

Then $0 \leq \lim\limits_{n \to +\infty} (u_n - 1) \leq 0$.

So $\lim\limits_{n} u_n = 1$

3

3

5) $\forall n \in \mathbb{N}$; $v_n = \dfrac{1}{U_n}$

,25 • (U_n) is decreasing.

,5 • $v_n = \dfrac{1}{U_n} \Rightarrow (v_n)$ is increasing or $(v_{n+1} - U_n = ..)$

• $\lim\limits_{n\to+\infty}(U_n - v_n) = \lim\limits_{n\to+\infty}\left(U_n - \dfrac{1}{U_n}\right)$

,5 $= \lim\limits_{n\to+\infty} \dfrac{U_n^2 - 1}{U_n} = \dfrac{1-1}{1} = 0$.

,25 Then (U_n) and (v_n) are adjacente.

(other methods are also accepts).

08

Exo3:

I

$$\lim_{x\to 0} \frac{e^x-1}{x(e^x+1)} = \frac{0}{0} \ (I.F).$$

using the L'Hopital's rule:

$$\lim_{x\to 0} \frac{e^x-1}{x(e^x+1)} = \lim_{x\to 0} \frac{(e^x-1)'}{[x(e^x+1)]'}$$

$$= \lim_{x\to 0} \frac{e^x}{e^x(1+x)+1} = \frac{1}{2}.$$

II/

$$f(x) = \begin{cases} \dfrac{e^x-1}{e^x+1} & ; \text{ if } x \leq 0 \\ ax & ; \text{ if } x > 0 \end{cases} \qquad a \in \mathbb{R}.$$

1) $\lim_{x \overset{<}{\to} 0} f(x) = f(0) = 0 = \lim_{x \overset{>}{\to} 0} ax$:

* f is Continuous at 0.

2) f is differentiable at 0 $\Longleftrightarrow$

$$\lim_{x \overset{>}{\to} 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \overset{<}{\to} 0} \frac{f(x)-f(0)}{x-0} = f'(0)$$

So:

0,25 $\lim_{x \overset{>}{\to} 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \overset{>}{\to} 0} \frac{ax}{x} = a.$

0,25 $\lim_{x \overset{<}{\to} 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \overset{<}{\to} 0} \frac{e^x-1}{(e^x+1)x} = \frac{1}{2}$

0,25 Then $a = \frac{1}{2} = f'(0)$.

1

$$f'(x) = \begin{cases} \dfrac{2e^x}{(e^x+1)^2} ; & \text{if } x \leq 0. \\ \dfrac{1}{2} ; & \text{if } x > 0. \end{cases}$$

1

3) * f is Continuous + * $f'(x) > 0$ $\Rightarrow$ f is bijective to $\mathbb{R}$ in $f(\mathbb{R})$.

x	$-\infty$	0	$+\infty$
$f(x)$	-1	0	$+\infty$

0,5 $f(\mathbb{R}) =]-1; +\infty[$ / $f(]-\infty; 0[) =]-1; 0[$

$f([0; +\infty[) = [0; +\infty[$

$\forall x \in \mathbb{R}, \forall y \in]-1; +\infty[$

0,25 $f(x) = y \Longleftrightarrow x = f^{-1}(y)$.

5

0,25 • $\frac{e^x - 1}{e^x + 1} = y \Leftrightarrow e^x - 1 = ye^x + y \Leftrightarrow e^x(-y+1) = y+1.$

$\Leftrightarrow e^x = \frac{y+1}{1-y} \Leftrightarrow x = \ln\left(\frac{y+1}{1-y}\right).$

0,25 • $\frac{1}{2}x = y \Leftrightarrow x = 2y$ . .

So:

0,25 $f^{-1}(x) = \begin{cases} \ln\left(\frac{x+1}{1-x}\right) & ; x \in]-1;0]. \\ 2x & ; x \in]0;+\infty[. \end{cases}$

0,25 $(f^{-1})'(0) = \frac{1}{f'(x_0)}$; with $f(x_0) = 0$.

0,25 $f(x_0) = 0 \Rightarrow x_0 = f^{-1}(0) = \ln\left(\frac{0+1}{1-0}\right) = 0$.

0,25 $f'(0) = 1/2$; So $(f^{-1})'(0) = 2$.

6