L'arbi Ben M'hidi University

Faculty: Exact Sciences, Natural and Life sciences **In:** 13/01/2026
Department : Mathematics and Computer Science. **Duration:** 1h30min
Level: Second year of Bachelor Mathematics

Mathematical logic exam

Exercice 1: Course questions
Prove the following propositions:
P1: *There exist two irrational numbers x and y such that x^y be rational.* $(2pts)$
P2: *Every integer greater than* 1 *is a product of primes* $(2pts)$
P3: *A strict order relation is antisymmetric.because there is no x and y* such that xRy and yRx. (R it means relation) $(2pts)$

Exercice 2:
Let A, B, P,Q and R , be five different propositions
1/ Use the truth table to determine whether $(\rceil P \vee Q) \wedge (Q \Longrightarrow (\rceil R \wedge \rceil P)) \wedge (P \vee R)$ is an antilogy. .$(3pts)$
2/ Show that If A and $(A \Longrightarrow B)$ are tautologies, then so is B.$(2pts)$
3/ Put $A : (P \Longrightarrow Q) \wedge (\rceil P \vee \rceil Q)$ and $B : P \Longleftrightarrow Q$
Write A and B using only the two logical connectors $\wedge$ and $\rceil$. $(2pts)$

Exercice 3: $(7pts)$
Ahmad, Basim and Camilia are three students that took the Logic exam. Let's consider a propositional language where
A ="Ahmad passed the exam",
B ="Basim passed the exam",
C ="Camilia passed the exam".
Formalize the following sentences:
1. "Camilia is the only one passing the exam"
2. "Ahmad is the only one not passing the exam"
3. "Only one, among Ahmad, Basim and Camilia, passed the exam"
4. "At least one among Ahmad, Basim and Camilia passed the exam"
5. "At least two among Ahmad, Basim and Camilia passed the exam"
6. "At most two among Ahmad, Basim and Camilia passed the exam"
7. "Exactly two, among Ahmad, Basim and Ccamilia passed the exam"

Correction

Exercice 1: Course questions

P1: *There exist two irrational numbers x and y such that x^y be rational.* (2*pts*)

Proof. We know that $\sqrt{2}$ is irrational. Consider the number $\sqrt{2}^{\sqrt{2}}$ which is either rational or irrational.

If $\sqrt{2}^{\sqrt{2}}$ is rational, the proposition is proved by considering $x = y = \sqrt{2}$.

If $\sqrt{2}^{\sqrt{2}}$ is irrational, then posing $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$, then we get $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$ and the proposition is proved. ■

P2: *Every integer greater than* 1 *is a product of primes* (2*pts*)

Proof. By contradiction and Well Ordering. Assume that the theorem is false and let C be the set of all integers greater than 1 that cannot be factored as a product of primes. We assume that C is not empty and derive a contradiction.

If C is not empty, there is a least element, $n_0 \in C$, by Well Ordering. The n_0 can't be prime, because a prime by itself is considered a product of primes (length one) and no such products are in C.

So n_0 must be a product of two integers a and b where $1 < a, b < n_0$. Since a and b are smaller than the smallest element in C, we know that $a, b \notin C$. In other words, a can be written as a product of primes $p_1.p_2...p_k$ and b as a product of primes $q_1.q_2..q_l$. Therefore, $n_0 = p_1.p_2...p_k.q_1.q_2..q_l$. can be written as a product of primes, contradicting the claim that $n_0 \in C$. Our assumption that C is not empty must therefore be false. ■

P3: *A strict order relation is antisymmetric.because there is no x and y such that xRy and yRx.* (2*pts*)

Proof. By definition R is transitive and anti-reflexive.

And we knew that a relation is antisymmetric if it satisfies $\forall x, y \in E : xRy \wedge yRx \Longrightarrow x = y$

We reason by absurdity.

Assume that there exists $x, y \in E$ such that the proposition $xRy \wedge yRx$ is true. Then by transitivity we get xRx is true, which contradicts the fact that R is irreflexive. Consequently, the proposition $xRy \wedge yRx$ is always false and so the logical implication $\forall x, y \in E : xRy \wedge yRx \Longrightarrow x = y$ is always true. ■

Exercice 2:

1/ put $(\rceil P \vee Q) \wedge (Q \Longrightarrow (\rceil R \wedge \rceil P)) \wedge (P \vee R)$ .... $(*)$

P	Q	R	$\rceil P$	$\rceil R$	$\rceil P \vee Q$	$\rceil R \wedge \rceil P$	$Q \Longrightarrow (\rceil R \wedge \rceil P)$	$P \vee R$	*proposition* $(*)$
1	1	1	0	0	1	0	0	1	0
1	1	0	0	1	1	0	0	1	0
1	0	1	0	0	0	0	1	1	0
1	0	0	0	1	0	0	1	1	0
0	1	1	1	0	1	0	0	1	0
0	1	0	1	1	1	1	1	0	0
0	0	1	1	0	1	0	1	1	1
0	0	0	1	1	1	1	1	0	0

So the given proposition is not an antilogy (3*pts*)

2/ Assume that A and $(A \Longrightarrow B)$ are tautologies. If B took the value F for some assignment of truth values to the statement letters of A and B, then, since A is a tautology, A would take the value T and, therefore, $(A \Longrightarrow B)$ would have the value F for that assignment. This contradicts the assumption that $(A \Longrightarrow B)$ is a tautology. Hence, B never takes the value F. $(2pts)$

3/ $A : (P \Longrightarrow Q) \wedge (\rceil P \vee \rceil Q)$

$\Leftrightarrow (\rceil P \vee Q) \wedge \rceil (P \wedge Q)$

$\Leftrightarrow \rceil (P \wedge \rceil Q) \wedge \rceil (P \wedge Q) \ldots . (1pt)$

$B : P \Longleftrightarrow Q$

$\Longleftrightarrow (P \Longrightarrow Q) \wedge (Q \Longrightarrow P)$

$\Longleftrightarrow (\rceil P \vee Q) \wedge (\rceil Q \vee P)$

$\Longleftrightarrow \rceil (P \wedge \rceil Q) \wedge \rceil (Q \wedge \rceil P) \quad (1pt)$

Exercice 3:

1/ $\overline{A} \wedge \overline{B} \wedge C .. (1pt)$

2/ $\overline{A} \wedge B \wedge C .. (1pt)$

3/ $\left(A \wedge \overline{B} \wedge \overline{C}\right) \vee \left(\overline{A} \wedge B \wedge \overline{C}\right) \vee \left(\overline{A} \wedge \overline{B} \wedge C\right) \ldots (1pt)$

4/ $\left(A \wedge \overline{B} \wedge \overline{C}\right) \vee \left(\overline{A} \wedge B \wedge \overline{C}\right) \vee \left(\overline{A} \wedge \overline{B} \wedge C\right) \vee \left(\overline{A} \wedge B \wedge C\right) \vee \left(A \wedge \overline{B} \wedge C\right) \vee \left(A \wedge B \wedge \overline{C}\right) \vee (A \wedge B \wedge C) \ldots (1pt)$

5/ $\left(\overline{A} \wedge B \wedge C\right) \vee \left(A \wedge \overline{B} \wedge C\right) \vee \left(A \wedge B \wedge \overline{C}\right) \vee (A \wedge B \wedge C) \ldots (1pt)$

6/ $\left(\overline{A} \wedge B \wedge C\right) \vee \left(A \wedge \overline{B} \wedge C\right) \vee \left(A \wedge B \wedge \overline{C}\right) \vee \left(A \wedge \overline{B} \wedge \overline{C}\right) \vee \left(\overline{A} \wedge B \wedge \overline{C}\right) \vee \left(\overline{A} \wedge \overline{B} \wedge C\right) \vee \left(\overline{A} \wedge \overline{B} \wedge \overline{C}\right) \ldots (1pt)$

7/ $\left(\overline{A} \wedge B \wedge C\right) \vee \left(A \wedge \overline{B} \wedge C\right) \vee \left(A \wedge B \wedge \overline{C}\right) \ldots (1pt)$