# OEB University – L3 Mathematics Exam

Module: Optimization

January 10, 2026

**Duration:** 90 minutes
**Mark:** /20

**Exercise 1** (Theory)

1. Prove that if the function $f : \mathbb{R}^n \to \mathbb{R}$ is *strictly convex*, then the minimization problem
$$\inf_{x \in \mathbb{R}^n} f(x)$$
admits at most one solution.
2. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is *not coercive* but still admits a global minimum.
3. Give an example of a function for which the first-order condition is satisfied at a point that is not a local minimum.
4. What is the rate of convergence? How is it related to the order of convergence?
5. Define a descent direction. Give an example in $\mathbb{R}^2$.

**Exercise 2** (Critical Points)
Consider the function
$$f(x, y) = x^3 + 2xy - 2x^2 - 2y^2.$$

1. Find the critical points of $f(x, y)$.
2. Determine the nature of each critical point.
3. Are the extremums local or global? Justify your answer.
4. Are the extremums unique? Justify your answer.

**Exercise 3** (Gradient Method)
Consider the function
$$f(x, y) = 2x^2 + 3y^2 - 2xy + 5x - 6.$$

(a) Compute the gradient $\nabla f(x, y)$.

(b) Determine a **descent direction** at $x_0 = (1, 2)$.

(c) Using a fixed step size $\beta = 0.1$, compute the iterations $x_1$ and $x_2$ of the gradient method.

# OEB University – L3 Mathematics Exam Solutions

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**Total Marks:** 20

**Exercise 1 (Theory) – 6 marks**

1. **Strictly convex function has at most one minimizer (2 marks):**
   If $f$ is strictly convex and $x_1 \neq x_2$ are minimizers, then for $\lambda \in (0, 1)$:

   $$f(\lambda x_1 + (1 - \lambda)x_2) < \lambda f(x_1) + (1 - \lambda)f(x_2) = f(x_1),$$

   which contradicts that $x_1$ and $x_2$ are minimizers. Hence, the minimizer is unique.

2. **Non-coercive function with global minimum (1 mark):** Example: $f(x) = x^2/(1 + x^2)$. *Reason:* $f$ is bounded below by 0, attains $\min f(x) = 0$ at $x = 0$, but $\lim_{|x|\to\infty} f(x) = 1$ so $f$ is not coercive.

3. **First-order condition satisfied but not a minimum (1 mark):** Example: $f(x) = x^3$ at $x = 0$. *Reason:* $f'(0) = 0$, but $x = 0$ is an inflection point, not a local minimum.

4. **Rate vs Order of Convergence (1 mark):** Rate $r$ measures how fast $|x_{k+1} - x^*| \approx r|x_k - x^*|^p$, where $p$ is the order of convergence. Higher $p$  faster convergence.

5. **Descent direction (1 mark):** Vector $d$ is a descent direction at $x$ if $\nabla f(x)^T d < 0$. Example in $\mathbb{R}^2$: $f(x, y) = x^2 + y^2$, at $(1, 1)$, $d = (-1, -1)$ is a descent direction because $\nabla f(1, 1) = (2, 2)$ and $(2, 2) \cdot (-1, -1) = -4 < 0$.

**Exercise 2 (Critical Points) – 7 marks**

Function: $f(x, y) = x^3 + 2xy - 2x^2 - 2y^2$

1. **Critical points (2 marks):**

   $$f_x = 3x^2 + 2y - 4x = 0 \quad , \quad f_y = 2x - 4y = 0 \Rightarrow y = x/2$$

   Substitute $y = x/2$ into $f_x = 0$:

   $$3x^2 + 2(x/2) - 4x = 3x^2 - 3x = 0 \Rightarrow x = 0 \text{ or } x = 1$$

   Then $y = 0$ or $y = 1/2$. **Critical points:** $(0, 0)$ and $(1, 1/2)$

2. **Nature of each point (2 marks):** Hessian: $H = \begin{pmatrix} 6x-4 & 2 \\ 2 & -4 \end{pmatrix}$ At $(0,0)$: $H = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}$, $\det H = 12 > 0$, trace$= -8 < 0 \Rightarrow$ local maximum. At $(1,1/2)$: $H = \begin{pmatrix} 2 & 2 \\ 2 & -4 \end{pmatrix}$, $\det H = -12 < 0 \Rightarrow$ saddle point.

3. **Local/Global extrema (1.5 mark):** $(0,0)$ is local maximum, $(1,1/2)$ is saddle no global minimum or maximum.

4. **Uniqueness of extrema (1.5 mark):** Maximum is unique at $(0,0)$; no global minimum exists.

**Exercise 3 (Gradient Method) – 7 marks**
Function: $f(x,y) = 2x^2 + 3y^2 - 2xy + 5x - 6$

(a) **Gradient (1 marks):**

$$\nabla f = \begin{pmatrix} 4x - 2y + 5 \\ 6y - 2x \end{pmatrix}$$

(b) **Descent direction at** $(1,2)$ **(2 mark):**

$$\nabla f(1,2) = \begin{pmatrix} 4-4+5 \\ 12-2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

A descent direction: $d = -\nabla f(1,2) = (-5,-10)$

(c) **Iterations with** $\beta = 0.1$ **(4 marks):**

$$x_1 = x_0 + \beta d = (1,2) + 0.1(-5,-10) = (0.5,1)$$
$$\nabla f(x_1) = \begin{pmatrix} 4*0.5 - 2*1 + 5 \\ 6*1 - 2*0.5 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$
$$x_2 = x_1 + \beta(-\nabla f(x_1)) = (0.5,1) + 0.1(-4,-5) = (0.1,0.5)$$