Model Answer + Grading Rubric

Answer to exercise 01 : (Comparison of queues : 08 Marks)

0.5

1. This system can be modeled as an $M/M/2$ since it has Poisson arrivals, exponential service times, two servers, a FIFO discipline (default), illimited capacity (default), illimited population (default). 0.5
2. Probability of Finding the DBMS Empty (P_0) :
   $\varrho = \frac{\lambda}{m\mu} = \frac{14}{2\times 17} = 0.4117647058$
   $\varrho < 1 \implies$ The system is stable.
   $P_0 = \left[\frac{(m\rho)^m}{m!(1-\rho)} + \sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!}\right]^{-1} = \left[\frac{(2\times 0.4117647058)^2}{2!(1-0.4117647058)} + 1 + 2\times 0.4117647058\right]^{-1}$
   $\boxed{P_0 = 0.4166666666}$ 1.0
3. Probability That a Request Has to Wait (ζ) : 0.5
   $\zeta = \frac{(m\rho)^m}{m!(1-\rho)} P_0 = \frac{(2\times 0.4117647058)^2}{2!(1-0.4117647058)} \times 0.4166666666$
   $\boxed{\zeta = 0.240196}$ 1.0
4. Average Number of Requests in the System $(\overline{N})$: 0.5
   $\overline{N} = \overline{Q} + \overline{R} = \frac{\zeta\varrho}{1-\rho} + m\varrho = \frac{0.240196\times 0.4117647058}{1-0.4117647058} + 2\times 0.4117647058$
   $\boxed{\overline{N} = 0.9916666666}$ 1.0
5. Tthe average residence time in this system $(\overline{T})$: 0.5
   Using Little's formula : $\overline{N} = \lambda\overline{T} \implies \overline{T} = \frac{\overline{N}}{\lambda} = \frac{0.9916666666}{14}$
   $\boxed{\overline{T} = 0.0708333333}$ 1.0
6. The probability of finding less than 2 requests in this system at a given time :
   $P_* = 1 - \zeta = 1 - 0.240196$ 0.5
   $\boxed{P_* = 0.759804}$ 1.0

Answer to exercise 02 : (Queueing network : 12 Marks)

Nous avons : $\gamma = 5, m_1 = +\infty, m_2 = 1, m_3 = 2, \mu_1 = \mu_2 = \mu_3 = \mu$.

1. Internal routing probability matrix : 0.5
   $\boxed{\begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix}}$, External routing probability matrix : $\boxed{\begin{pmatrix} 0 \\ \frac{3}{4} \\ \frac{3}{4} \end{pmatrix}}$ 0.5
2. Effective arrival rates λ_i :
   $$\begin{cases} \lambda_1 = \gamma + \frac{1}{5}\lambda_1 + \frac{1}{4}\lambda_2 + \frac{1}{4}\lambda_3 \\ \lambda_2 = \frac{2}{5}\lambda_1 \\ \lambda_3 = \frac{2}{5}\lambda_1 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 5 + \frac{2}{5}\lambda_1 \\ \lambda_2 = \lambda_3 = \frac{2}{5}\lambda_1 \end{cases} \Rightarrow \begin{cases} \frac{3}{5}\lambda_1 = 5 \\ \lambda_2 = \lambda_3 = \frac{2}{5}\lambda_1 \end{cases}$$
   $$\Rightarrow \begin{cases} \lambda_1 = \frac{25}{3} \\ \lambda_2 = \lambda_3 = \frac{2}{5}\lambda_1 \end{cases} \Rightarrow \boxed{\begin{cases} \lambda_1 = \frac{25}{3} \\ \lambda_2 = \frac{10}{3} \\ \lambda_3 = \frac{10}{3} \end{cases}}$$ 0.5
3. The servers' utilization ϱ_i as a function of μ.
   $$\begin{cases} \varrho_1 = 0 \\ \varrho_2 = \frac{\lambda_2}{m_2\mu_2} \\ \varrho_3 = \frac{\lambda_3}{m_3\mu_3} \end{cases} \Rightarrow \begin{cases} \varrho_1 = 0 \\ \varrho_2 = \frac{10}{3\mu} \\ \varrho_3 = \frac{10}{3\times 2\times\mu} \end{cases} \Rightarrow \boxed{\begin{cases} \varrho_1 = 0 \\ \varrho_2 = \frac{10}{3\mu} \\ \varrho_3 = \frac{5}{3\mu} \end{cases}}$$ 0.5
4. Values of μ which ensure the stability of the network :
   $$\begin{cases} FA_1 \text{ Stable} \\ \wedge \\ FA_2 \text{ Stable} \\ \wedge \\ FA_3 \text{ Stable} \end{cases} \Rightarrow \begin{cases} \varrho_1 = 0 < 1 \\ \wedge \\ \varrho_2 = \frac{10}{3\mu} < 1 \\ \wedge \\ \varrho_3 = \frac{5}{3\mu} < 1 \end{cases} \Rightarrow \begin{cases} \mu > \frac{10}{3} \\ \wedge \\ \mu > \frac{5}{3} \end{cases}$$
   network stable $\implies \mu > \frac{10}{3}$

We conclude that for the network to be stable, μ must verify the following condition :

$\boxed{\mu > \frac{10}{3}}$ or $\boxed{\mu \in]\frac{10}{3}, +\infty[}$ **0.5**

For $\mu = 4$:

$$\begin{cases} \varrho_1 = 0 \\ \varrho_2 = \frac{\lambda_2}{m_2\mu_2} = \frac{10}{3\times 4} \\ \varrho_3 = \frac{\lambda_3}{m_3\mu_3} = \frac{5}{3\times 4} \end{cases} \Rightarrow \boxed{\begin{cases} \varrho_1 = 0 \\ \varrho_2 = \frac{5}{6} = 0.8333333333 \\ \varrho_3 = \frac{5}{12} = 0.4166666666 \end{cases}}$$ **0.5**

Queue 1 : of type $M/M/+\infty$. **Queue 2 :** de type $M/M/2$. **Queue 3 :** de type $M/M/1$.

5. The average number of customers waiting in each queue and in the network :
   (a) $\boxed{\overline{Q_1} = 0}$ **0.5**
   (b) $\overline{Q_2} = \frac{\varrho_2^2}{1-\varrho_2} = \frac{0.8333333333^2}{1-0.8333333333} \Longrightarrow \boxed{\overline{Q_2} = 4.1666666666}$ **0.5**
   (c) $P_0(3) = \left[\frac{(m_3\rho_3)^{m_3}}{m_3!(1-\rho_3)} + \sum_{k=0}^{m_3-1}\frac{(m_3\rho_3)^k}{k!}\right]^{-1} = \left[\frac{(2\times 0.4166666666)^2}{2!(1-0.4166666666)} + 1 + 2\times 0.4166666666\right]^{-1}$
   $\boxed{P_0(3) = 0.4117647058}$
   $\zeta_3 = \frac{(m_3\rho_3)^{m_3}}{m_3!(1-\rho_3)}P_0(3) = \frac{(2\times 0.4166666666)^2}{2!(1-0.4166666666)} \times 0.4166666666 \Rightarrow \boxed{\zeta_3 = 0.2450980392}$
   $\overline{Q_3} = \frac{\zeta_3\rho_3}{1-\rho_3} = \frac{0.2450980392\times 0.4166666666}{1-0.4166666666} \Rightarrow \boxed{\overline{Q_3} = 0.1750700280}$ **0.5**
   (d) $\overline{Q_R} = \sum_{i=1}^{3}\overline{Q_i} = 0 + 4.1666666666 + 0.1750700280 \Rightarrow \boxed{\overline{Q_R} = 4.3417366946}$ **0.5**
6. The average number of customers in each queue and in the network :
   (a) $\overline{N_1} = \overline{R_1} = \frac{\lambda_1}{\mu_1} \Rightarrow \boxed{\overline{N_1} = \frac{25}{12} = 2.0833333333}$ **0.5**
   (b) $\overline{N_2} = \overline{Q_2} + \overline{R_2} = 4.1666666666 + 0.8333333333 \Rightarrow \boxed{\overline{N_2} = 5}$ **0.5**
   (c) $\overline{N_3} = \overline{Q_3} + \overline{R_3} = 0.1750700280 + 2\times 0.4166666666 \Rightarrow \boxed{\overline{N_3} = 1.00840336135}$ **0.5**
   (d) $\overline{N_R} = \sum_{i=1}^{3}\overline{N_i} = 2.0833333333 + 5 + 1.00840336135 \Rightarrow \boxed{\overline{N_R} = 8.0917366946}$ **0.5**
7. The average residence time in each queue and in the network :
   (a) $\overline{T_1} = \frac{1}{\mu_1} = \frac{1}{4} \Rightarrow \boxed{\overline{T_1} = 0.25}$ **0.5**
   (b) $\overline{T_2} = \frac{\overline{N_2}}{\lambda_2} = \frac{5}{\frac{10}{3}} \Rightarrow \boxed{\overline{T_2} = \frac{3}{2} = 1.5}$ **0.5**
   (c) $\overline{T_3} = \frac{\overline{N_3}}{\lambda_3} = \frac{1.00840336135\times 3}{10} \Rightarrow \boxed{\overline{T_3} = 0.3025210084}$ **0.5**
   (d) $\overline{T_R} = \frac{\overline{N_R}}{\lambda_R} = \frac{\overline{N_R}}{\sum_{i=1}^{2}\gamma_i} = \frac{8.0917366946}{5} \Rightarrow \boxed{\overline{T_R} = 1.6183473389}$ **0.5**
8. The average waiting time in each queue and in the network :
   (a) $\boxed{\overline{W_1} = 0}$ **0.5**
   (b) $\overline{W_2} = \frac{\overline{Q_2}}{\lambda_2} = \frac{4.1666666666\times 3}{10} \Rightarrow \boxed{\overline{W_2} = 1.25}$ **0.5**
   (c) $\overline{W_3} = \frac{\overline{Q_3}}{\lambda_3} = \frac{0.1750700280\times 3}{10} \Rightarrow \boxed{\overline{W_3} = 0.0525210084}$ **0.5**
   (d) $\overline{W_R} = \frac{\overline{Q_R}}{\lambda_R} = \frac{\overline{Q_R}}{\sum_{i=1}^{2}\gamma_i} = \frac{0+\frac{1}{2}+\frac{1}{12}}{4} \Rightarrow \boxed{\overline{W_R} = 0.8683473389}$ **0.5**
9. The probability that the network is empty :
   $Pr(\text{Network empty}) = Pr(\text{Queue}_1 \text{ empty and Queue}_2 \text{ empty and Queue}_3 \text{ empty})$
   $= P_0(\text{Queue}_1) \times P_0(\text{Queue}_2) \times P_0(\text{Queue}_3)$
   $Pr(\text{Network not empty}) = e^{\frac{-\lambda_1}{\mu_1}} \times P_0(2) \times P_0(3) = e^{-\frac{25}{12}} \times (1-\rho_2) \times P_0(3)$
   $= 0.1245144714 \times 0.1666666666 \times 0.4117647058 = 0.0085451107$
   Hence : $\boxed{Pr(\text{Network not empty}) = 0.0085451107}$ **1.0**