L'arbi Ben M'hidi University

Faculty: Exact sciences and sciences of nature and life
Departement: MI
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Module: Algebra 2

Correction of Exam n 2

<u>**Exercise 1:**</u>(8 **pts**)
1.

$$F_1 = \left\{(x, y, z) \in \mathbb{R}^3 / 3x - y + z = 0\right\}$$

i. Let $(x, y, z), \left(x^{'}, y^{'}, z^{'}\right)$ 2 elements of F_1. So,

$$\left\{ \begin{array}{c} 3x - y + z = 0 \\ 3x^{'} - y^{'} + z^{'} = 0 \end{array} \right. \Rightarrow 3\left(x + x^{'}\right) - \left(y + y^{'}\right) + \left(z + z^{'}\right) = 0$$

So $(x, y, z) + \left(x^{'}, y^{'}, z^{'}\right) = \left(x + x^{'}, y + y^{'}, z + z^{'}\right) \in F_1$ ....(1 **pts**)
ii. Likewise,
$\forall (x, y, z) \in F_1$, $\forall \lambda \in \mathbb{R}$ we have $\lambda (x, y, z) = (\lambda x, \lambda y, \lambda z) \in F_1$ ....(1 **pts**)
because

$$3\lambda x - \lambda y + \lambda z = \lambda (3x - y + z) = 0$$

so, the set F_1 is a vector subspace.
2.

$$F_2 = \left\{(x, y) \in \mathbb{R}^2 / 2x - y = 0\right\}$$

i. Let $(x, y), \left(x^{'}, y^{'}\right)$ 2 elements of F_2. So,

$$\left\{ \begin{array}{c} 2x - y = 0 \\ 2x^{'} - y^{'} = 0 \end{array} \right. \Rightarrow 2\left(x + x^{'}\right) - \left(y + y^{'}\right) = 0$$

$\Rightarrow (x, y) + \left(x^{'}, y^{'}\right) = \left(x + x^{'}, y + y^{'}\right) \in F_2$ ....(1 **pts**)
ii. Likewise,
$\forall (x, y) \in F_1$, $\forall \lambda \in \mathbb{R}$ on a $\lambda (x, y) = (\lambda x, \lambda y) \in F_2$ ....(1 **pts**)
Likewise,

$$2\lambda x - \lambda y = \lambda (2x - y) = 0$$

so, the set F_2 is a vector subspace.
3. Find a basis of F_1

$$\begin{aligned} F_1 &= \left\{(x, y, z) \in \mathbb{R}^3 / 3x - y + z = 0\right\} \\ &= \{(x, 3x + z, z) / x, z \in \mathbb{R}\} \\ &= \{x (1, 3, 0) + z (0, 1, 1) / x, z \in \mathbb{R}\} \end{aligned}$$

Since F_1 is generated by the 2 vectors $\{(1,3,0),(0,1,1)\}$ which are linearly idependent, ...(1 **pts**)

so $\{(1,3,0),(0,1,1)\}$ form a basis of F_1 and its dimension is 2.....................(1 **pts**)

4. Find a basis of F_2

$$\begin{aligned} F_2 &= \left\{(x,y) \in \mathbb{R}^2 / 2x - y = 0\right\} \\ &= \{(x,2x)/x \in \mathbb{R}\} \\ &= \{x(1,2)/x \in \mathbb{R}\} \end{aligned}$$

Since F_1 is generated by $\{(1,2)\}$ which is non-zero, ...(1 **pts**)

so $\{(1,2)\}$ form a basis of F_2 and its dimension is 1.....................(1 **pts**)

Exercise 2:(5 **pts**)

Let the following vectors

$$v_1 = (0,1,-2), v_2 = (1,1,0), v_3 = (-2,0,-2)$$

1. Prove that v_1, v_2, v_3 form a basis of $\mathbb{R}^3$.

we have

$$av_1 + bv_2 + cv_3 = (0,0,0) \Rightarrow (b-2c, a+b, -2a-2c) = (0,0,0,) \Rightarrow a = b = c = 0$$

then, $\{v_1, v_2, v_3\}$ are linearly idependent, ...(1 **pts**)

because the dimension of $\mathbb{R}^3$ is 3, then $\{v_1, v_2, v_3\}$ form a basis of $\mathbb{R}^3$.....................(1 **pts**)

2. Determine the coordinates of u_1, u_2 in the basis $\{v_1, v_2, v_3\}$ where

$$\begin{aligned} u_1 &= 7v_2 - 5v_3 \\ u_2 &= (-2,6,0) \end{aligned}$$

i. the coordinates of u_1 in the basis $\{v_1, v_2, v_3\}$ are $(0,7,-5)$ ....(1.5 **pts**)

ii. Let (a,b,c) the coordinates of u_2 in the basis $\{v_1, v_2, v_3\}$, so

$$\begin{aligned} u_2 &= av_1 + bv_2 + cv_3 \\ &\Rightarrow (-2,6,0) = a(0,1,-2) + b(1,1,0) + c(-2,0,-2) \\ &\Rightarrow a = -4, b = 10, c = 4. \end{aligned}$$

then, the coordinates of u_2 in the basis $\{v_1, v_2, v_3\}$ are $(-4,10,4)$ ....(1.5 **pts**)

Exercise 3:(7 **pts**)

Let the matrix :

$$P = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

(1) Find a and b where

$$P^2 + aP + bI_2 = 0_2$$

We have

$$\begin{aligned}
P^2 + aP + bI_2 &= 0_2 \\
&\Rightarrow \begin{pmatrix} 4 & 0 \\ 5 & 9 \end{pmatrix} + a\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&\Rightarrow \begin{cases} 2a + b + 4 = 0 \\ a + 5 = 0 \\ 3a + b + 9 = 0 \end{cases} \\
&\Rightarrow a = -5, b = 6. \qquad (\text{...................}(2 \textbf{ pts}))
\end{aligned}$$

(2) Deduce that P is an invertible matrix and calculate P^{-1}
We have

$$P^2 - 5P + 6I_2 = 0_2 \Rightarrow P\left(\frac{5}{6}I_2 - P\right) = I_2 \qquad (\text{...................}(1 \textbf{ pts}))$$

so P is invertible and $P^{-1} = \frac{5}{6}I_2 - \frac{1}{6}P = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$....................(1 **pts**)

(3) Let the linear application $f : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$f(x, y) = (2x, x + 3y)$$

Prove that f^{-1} exists and find its formula.

We have $f(1, 0) = (2, 1), f(0, 1) = (0, 3)$, so the matrix of f according to the canonical basis is P....................(1 **pts**)

Because P is invertible, so f is bijectiv and f^{-1} exists and $P^{-1}$is the matrix associated of f^{-1}....................(1 **pts**)

Then,

$$\begin{aligned}
f^{-1}(x, y) &= xf^{-1}(1, 0) + yf^{-1}(0, 1) \\
&= x(\frac{1}{2}, -\frac{1}{6}) + y(0, \frac{1}{3}) \\
&= (\frac{1}{2}x, -\frac{1}{6}x + \frac{1}{3}y) \qquad (\text{...................}(1 \textbf{ pts}))
\end{aligned}$$

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