L'arbi Ben M'hidi University

Faculty: Exact sciences and sciences of nature and life
Departement: MI
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Module: Algebra 2

Correction of Exam n 2

Exercise 1:(5 **pts**)
1. Prove that F_1 is a vector subspace

$$\Leftrightarrow \begin{cases} 1)\ F_1 \neq \varnothing \\ 2) \forall (x,y), \left(x^{'}, y^{'}\right) \in F_1, (x,y) + \left(x^{'}, y^{'}\right) \in F_1 \\ 3) \forall (x,y) \in F_1, \forall \lambda \in \mathbb{R}, \lambda (x,y) \in F_1 \end{cases} \text{................................}(1.5\ \textbf{pts})$$

$$F_1 = \left\{(x,y) \in \mathbb{R}^2 / 3x + y = 0\right\}$$

i. We have $F_1 \neq \varnothing$ because $(0,0) \in F_1$. ....(0.5 **pts**)
ii. Let $(x,y), \left(x^{'}, y^{'}\right)$ two elements of F_1. So,

$$\begin{cases} 3x + y = 0 \\ 3x^{'} + y^{'} = 0 \end{cases} \Rightarrow 3\left(x + x^{'}\right) + \left(y + y^{'}\right) = 0$$

so

$$(x,y) + \left(x^{'}, y^{'}\right) = \left(x + x^{'}, y + y^{'}\right) \in F_1 \text{....................}(1\ \textbf{pts})$$

iii. Likewise,
$\forall (x,y) \in F_1$, $\forall \lambda \in \mathbb{R}$ we have $\lambda (x,y) = (\lambda x, \lambda y) \in F_1$
because

$$3\lambda x + \lambda y = \lambda (3x + y) = 0 \text{...........}(1\ \textbf{pts})$$

so, F_1 is a vector subspace.
2.

$$F_2 = \left\{(x,y) \in \mathbb{R}^2 / y \leq 0\right\}$$

F_2 is not a vector subspace because if we take $(x,y) \in F_2$ with $y \prec 0$
and $\lambda \succ 0$, we have $\lambda y \succ 0$, which implies that $\lambda (x,y) \notin F_2$................................(1.0 **pts**)
Exercise 2:(8 **pts**)

$$f(x,y,z) = (-3x - y + z, 8x + 3y - 2z, -4x - y + 2z)$$

1. Prove that f is a linear application

$$\Longleftrightarrow \begin{cases} 1)\ \forall (x,y,z), \left(x^{'}, y^{'}, z^{'}\right) \in \mathbb{R}^3 : f\left[(x,y,z) + \left(x^{'}, y^{'}, z^{'}\right)\right] = f(x,y,z) + f\left(x^{'}, y^{'}, z^{'}\right) \\ 2)\ \forall (x,y,z) \in \mathbb{R}^3, \forall \lambda \in \mathbb{R} : f[\lambda (x,y,z)] = \lambda f(x,y,z) \end{cases} \text{.......}(1.0\ \textbf{pts})$$

i. $\forall (x, y, z), \left(x^{'}, y^{'}, z^{'}\right) \in \mathbb{R}^3$:

$$\begin{aligned} & f\left[(x,y,z)+\left(x^{'},y^{'},z^{'}\right)\right] \\ = & f\left(x+x^{'},y+y^{'},z+z^{'}\right) \\ = & \left(-3\left(x+x^{'}\right)-\left(y+y^{'}\right)+\left(z+z^{'}\right),8\left(x+x^{'}\right)+3\left(y+y^{'}\right)-2\left(z+z^{'}\right),-4\left(x+x^{'}\right)-\left(y+y^{'}\right)+2\right. \\ = & (-3x-y+z,8x+3y-2z,-4x-y+2z)+\left(-3x^{'}-y^{'}+z^{'},8x^{'}+3y^{'}-2z^{'},-4x^{'}-y^{'}+2z^{'}\right) \\ = & f(x,y,z)+f\left(x^{'},y^{'},z^{'}\right) \text{......................} (0.75\ \textbf{pts}) \end{aligned}$$

ii. $\forall (x, y, z) \in \mathbb{R}^3$, $\forall \lambda \in \mathbb{R}$ we have

$$\begin{aligned} f[\lambda(x,y,z)] &= f(\lambda x,\lambda y,\lambda z) \\ &= (-3\lambda x-\lambda y+\lambda z,8\lambda x+3\lambda y-2\lambda z,-4\lambda x-\lambda y+2\lambda z) \\ &= \lambda(-3x-y+z,8x+3y-2z,-4x-y+2z) \\ &= \lambda f(x,y,z) \text{................} (0.75\ \textbf{pts}) \end{aligned}$$

2. Find a basis of $\ker f$ and its dimension.

$$\ker f = \left\{(x,y,z) \in \mathbb{R}^3 / f(x,y,z) = (0,0,0)\right\} \text{.....} (1\ \textbf{pts})$$

so,

$$\begin{cases} -3x-y+z=0 \\ 8x+3y-2z=0 \\ -4x-y+2z=0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} z=x \\ y=-2x \end{cases}$$

$$\Rightarrow \quad \ker f = \{(x,-2x,x)/x \in \mathbb{R}\} = \{x(1,-2,1)/x \in \mathbb{R}\} \text{...........} (1\ \textbf{pts})$$

so, $\{(1,-2,1)\}$ is a basis of $\ker f$ .and $dim(\ker f) = 1$............ (1.0 **pts**)

3. f is not injective because

$$\ker f \neq \{(0,0,0)\} \text{..............} (1\ \textbf{pts})$$

4. The Rank of f.

we have

$$\begin{aligned} \dim \mathbb{R}^3 &= rgf + dim(\ker f) \text{........} (0.5\ \textbf{pts}) \\ &\Rightarrow rgf = 3-1 = 2 \text{..........} (0.5\ \textbf{pts}) \end{aligned}$$

Because $dim\ \mathrm{Im}\, f = rgf = 2 \prec dim\mathbb{R}^3 = 3 \Rightarrow \mathrm{Im}\, f \neq \mathbb{R}^3 \Rightarrow f$ is not surjective.......... (0.5 **pts**)

Exercice 3:(7 **pts**)

$A = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 7 & -5 \\ -3 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 1 & 1 \\ 1 & -3 & 3 \\ 2 & 3 & 1 \end{pmatrix}$,

1) $AB = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix} \times \begin{pmatrix} 7 & -5 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$............. (1.0 **pts**)

We can not calculate AC because the number of de columns of $A \neq$ the number of lines of C ....(1 **pts**)

2) Is C invertible ?

$detC = 3 \begin{vmatrix} -3 & 3 \\ 3 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix} = -22$..............(1.5 **pts**)

Because $detC = -22 \neq 0$ so C is invertible ....(1 **pts**)

3) Find C^{-1}.

$C^{-1} = \dfrac{1}{detC} (Adj)^t$ (..........0.5 **pts**)

$C^{-1} = \dfrac{1}{detC} \begin{pmatrix} -12 & 5 & 9 \\ 2 & 1 & -7 \\ 6 & -8 & -10 \end{pmatrix}^t \Rightarrow \dfrac{1}{-22} \begin{pmatrix} -12 & 2 & 6 \\ 5 & 1 & -8 \\ 9 & -7 & -10 \end{pmatrix} = \begin{pmatrix} \frac{6}{11} & \frac{-1}{11} & \frac{-3}{11} \\ \frac{-5}{22} & \frac{-1}{22} & \frac{4}{11} \\ \frac{-9}{22} & \frac{7}{22} & \frac{5}{11} \end{pmatrix}$........... (2 **pts**)

PR. Rezzag.S