University of Oum El Bouaghi
Faculty of SENV, Department of MI
Final Exam in Algebra 4, 2024/ 2025
Time 1H.30m

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Exercise 1 (14 marks)
Let

$$A = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}$$

1. For what values of a, the matrix A represents a scalar product?
2. Write the diagonal form of A.
3. Determine the orthogonal for the bilinear form φ_A of $F = vect\{(1,1,1)\}$.
4. For what values of a that $F \oplus F^{\perp} = \mathbb{R}^3$ holds.
5. According to the values of a, determine the orthogonal of $\mathbb{R}^3$

Exercise 2 (6 marks)

1. Prove that, if u and v are nonzero orthogonal vectors, then they are free. Show that the converse is not right.
2. Prove that the eigenvalues of a real symmetric matrix are real.

Correction

Solution of exercise 1 (14 marks)

1. (3 marks) A scalar product is a symmetric positive definite bilinear form. That means, it is a symmetric bilinear form with strict positive coefficients in the diagonal form, or equivalently, the eigenvalues of the symmetric matrix are all strictly positive.

   Since the matrix A is symmetric, then it represents a bilinear form φ_A. So we have to find the eigenvalues of A:

   $$\det(A - \lambda I) = 0 \Rightarrow (X - a - 2)(X - a + 1)^2 = 0,$$

   which gives

   $$\lambda_1 = a + 2,\ \lambda_2 = \lambda_3 = a - 1$$

   Therefore,

   $$\lambda_1 > 0 \text{ and } \lambda_2 = \lambda_3 > 0 \text{ for } a > 1.$$

2. (2 marks) The diagonal form of φ_A is

   $$\varphi_A(X) = (a + 2)x^2 + (a - 1)y^2 + (a - 1)z^2$$

   where

   $$X = xv_1 + yv_2 + zv_3$$

   with $\{v_1, v_2, v_3\}$ is the basis of the eigenvectors of the matrix A.

3. (3 marks) The orthogonal for the bilinear form φ_A of $F = vect\{V = (1, 1, 1)\}$ is

   $$F^\perp = \left\{X \in \mathbb{R}^3, \varphi_A(X, V) = 0\right\}$$

   First, let us write the bilinear form φ_A:

   $$\varphi_A(X, Y) = ax_1y_1 + x_1y_2 + x_1y_3 + x_2y_1 + ax_2y_2 + x_2y_3 + x_3y_1 + x_3y_2 + ax_3y_3 \tag{1}$$

   where

   $$X = (x_1, x_2, x_3),\ Y = (y_1, y_2, y_3)$$

Then, it is sufficient to replace Y by the vector $V = (1, 1, 1)$ in (1), we get

$$\begin{aligned} 0 &= \varphi_A(X, V) = ax_1 + x_1 + x_1 + x_2 + ax_2 + x_2 + x_3 + x_3 + ax_3 \\ &= (a+2)x_1 + (a+2)x_2 + (a+2)x_3 \\ &= (a+2)(x_1 + x_2 + x_3) \end{aligned}$$

Thus, for $a \neq -2$, we have

$$x_1 + x_2 + x_3 = 0$$

that means, the orthogonal of F is

$$F^{\perp} = \left\{(x, y, z) \in \mathbb{R}^3, x + y + z = 0\right\}$$

or equivalently

$$F^{\perp} = vect\left\{(1, 0, -1), (0, 1, -1)\right\}.$$

While for $a = -2$, we have

$$F^{\perp} = \mathbb{R}^3.$$

4. (3 marks) From the answer in 3, for $a \neq -2$, we have $F^{\perp} = vect\left\{(1, 0, -1), (0, 1, -1)\right\}$. That gives the set

$$\left\{(1, 0, -1), (0, 1, -1), (1, 1, 1)\right\} = \left\{(1, 0, -1), (0, 1, -1)\right\} \cup \left\{(1, 1, 1)\right\}$$

is a basis of $\mathbb{R}^3$ because $\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} = 3 \neq 0$. Therefore, we have

$$a \neq -2 \Rightarrow F \oplus F^{\perp} = \mathbb{R}^3.$$

5. (3 marks) Since the orthogonal of $\mathbb{R}^3$ is $\ker \varphi_A = \ker A \Rightarrow \det(A - 0 \times I) = 0$. That means, the orthogonal of $\mathbb{R}^3$ is associated to the eigenvalue $\lambda = 0$. Since $\lambda_1 = a + 2$, $\lambda_2 = \lambda_3 = a - 1$, then, $a = -2$ or $a = 1$.

   For $a = -2$, we get

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

That gives

$$\left(\mathbb{R}^3\right)^\perp = \ker A = vect\left\{(1,1,1)\right\} = F$$

For $a = 1$, we get

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

That gives

$$\left(\mathbb{R}^3\right)^\perp = \ker A = vect\left\{(-1,1,0), (-1,0,1)\right\}.$$

<u>Solution of exercise 2</u> (6 marks=3+3)

1. Let α, $\beta \in \mathbb{R}$ such that

   $$\alpha u + \beta v = 0$$

   Then, we have

   $$\langle \alpha u + \beta v, u\rangle = 0 \text{ and } \langle \alpha u + \beta v, v\rangle = 0$$

   That gives

   $$\alpha \langle u, u\rangle + \beta \langle u, v\rangle = 0 \text{ and } \alpha \langle u, v\rangle + \beta \langle v, v\rangle = 0$$

   Since u and v are nonzero orthogonal vectors, that gives $\langle u, u\rangle \neq 0$, $\langle v, v\rangle \neq 0$ and $\langle u, v\rangle = 0$. Therefore, $\alpha = \beta = 0$.

2. Let λ be an eigenvalue of a real symmetric matrix A associated to an eigenvector v. Then

   $$\langle Av, v\rangle = \langle \lambda v, v\rangle = \lambda \langle v, v\rangle \tag{1}$$

   In the other hand,

   $$\langle Av, v\rangle = \left\langle v, A^t v\right\rangle = \langle v, Av\rangle = \langle v, \lambda v\rangle = \overline{\lambda} \langle v, v\rangle$$

   Since v is an eigenvector, that means $v \neq 0$, which yields $\langle v, v\rangle \neq 0$. Therefore, $\lambda = \overline{\lambda}$, which means $\lambda \in \mathbb{R}$.