Model Answer + Grading Rubric

Answer to exercise 01 : (COMPARISON OF QUEUES : 05 Marks)

We have $\varrho_1 = \varrho_2 = \varrho = \frac{\lambda}{\mu}$.

1. Probability of an Empty System :

   $\boxed{P_{0M/M/1} = 1 - \varrho}$ ← **0.5**

   $P_{0M/M/2} = \left[\frac{(2\varrho)^2}{2!(1-\varrho)} + 1 + 2\varrho\right]^{-1} = \left[\frac{2\varrho^2}{1-\varrho} + \frac{1-\varrho}{1-\varrho} + \frac{2\varrho(1-\varrho)}{1-\varrho}\right]^{-1} = \left[\frac{2\varrho^2+1-\varrho+2\varrho(1-\varrho)}{1-\varrho} +\right]^{-1} = \left[\frac{1+\varrho}{1-\varrho}\right]^{-1}$

   $\boxed{P_{0M/M/2} = \frac{1-\varrho}{1+\varrho}}$. ← **0.5**

2. We notice that $P_{0M/M/2} = \frac{P_{0M/M/1}}{1+\varrho}$. $\varrho > 0 \Rightarrow 1 + \varrho > 1$. Hence, $\boxed{P_{0M/M/2} < P_{0M/M/1}}$ ← **0.5**

3. Average Number of Waiting Customers in the System :

   $\boxed{\overline{Q}_{M/M/1} = \frac{\varrho^2}{1-\varrho}}$ ← **0.5**

   $\zeta = \frac{2\varrho^2}{1-\varrho}\frac{1-\varrho}{1+\varrho} = \frac{2\varrho^2}{1+\varrho}$. So, $\overline{Q}_{M/M/2} = \frac{\zeta\varrho}{1-\varrho} = \frac{2\varrho^2\varrho}{(1-\varrho)(1+\varrho)}$. Hence, $\boxed{\overline{Q}_{M/M/2} = \frac{2\varrho^3}{1-\varrho^2}}$ ← **1.0**

4. We notice that $\overline{Q}_{M/M/2} = \frac{\varrho}{1+\varrho}\overline{Q}_{M/M/1}$. Since $\varrho < 1 + \varrho \Rightarrow \frac{\varrho}{1+\varrho} < 1$. We conclude that $\boxed{\overline{Q}_{M/M/2} < \overline{Q}_{M/M/1}}$ ← **0.5**

5. Average Number of Customers in the System :

   $\boxed{\overline{N}_{M/M/1} = \frac{\varrho}{1-\varrho}}$ ← **0.5**

   $\overline{N}_{M/M/2} = \overline{Q}_{M/M/2} + \overline{R}_{M/M/2} = \frac{2\varrho^3}{1-\varrho^2} + 2\varrho = \frac{2\varrho^3+2\varrho(1-\varrho^2)}{1-\varrho^2}$. Hence, $\boxed{\overline{N}_{M/M/2} = \frac{2\varrho}{1-\varrho^2}}$ ← **0.5**

6. We notice that $\overline{N}_{M/M/2} = \frac{2}{1+\varrho}\overline{N}_{M/M/1}$. We have $\varrho < 1 \Rightarrow 1 + \varrho < 2 \Rightarrow \frac{2}{1+\varrho} > 1$.

   We conclude that $\boxed{\overline{N}_{M/M/2} > \overline{N}_{M/M/1}}$ ← **0.5**

Answer to exercise 02 : (QUEUEING SYSTEM : 05 Marks)

1. We have $\lambda = 5$ requests/s and $\mu = 6$ requests/s. The appropriate model for the system is $M/M/1/N$ because : ← **1.0**
   (a) The arrival process is Poisson (statement).
   (b) Exponential service time (statement).
   (c) The number of servers is $m = 1$ (statement).
   (d) Service discipline is $FIFO$ (default).
   (e) Finite capacity (N requests in the system/limited memory size).
   (f) Infinite population size (default).

   } ← **1.0**

2. The minimal memory space required to reach the objective, if each request needs 1500 Bytes is :

   $\boxed{\varrho = \frac{\lambda}{\mu} = \frac{5}{6} \approx 0,83333333 \neq 1}$, $P_0 = \frac{1-\varrho}{1-\varrho^{c+1}}$, $P_N = \frac{(1-\varrho)\varrho^N}{1-\varrho^{N+1}}$

   The effective request rate accepted by the server is λ_e. To satisfy at least 90% of the received requests, $\frac{\lambda_e}{\lambda} \geq 0.9$ must hold.

   $\frac{\lambda_e}{\lambda} \geq 0.9 \implies \frac{\lambda(1-P_N)}{\lambda} \geq 0.9 \implies 1 - P_N \geq 0.9 \implies 1 - \frac{(1-\varrho)\varrho^N}{1-\varrho^{N+1}} \geq 0.9 \implies \frac{1-\varrho^N}{1-\varrho^{N+1}} \geq 0.9$

   Hence : $\frac{\lambda_e}{\lambda} \geq 0.9 \implies (1 - 0.9\varrho)\varrho^N \leq 0.1 \implies \varrho^N \leq \frac{0.1}{1-0.9\varrho} \implies \varrho^N \leq 0,4 \implies N\ln(\varrho) \leq \ln(0,4)$

   $\frac{\lambda_e}{\lambda} \geq 0.9 \implies N\ln(\frac{5}{6}) \leq \ln(0,4) \implies N \geq \frac{\ln(0,4)}{\ln(\frac{5}{6})}$ ($\ln(\varrho) < 0$ because $\varrho < 1$)

   Finally, $N \geq 5,025685101562$ and at least $\boxed{N = 6}$. Nedded memory = $\boxed{N \times 1500 = 9600 Bytes}$. ← **1.0**

3. The average number of daily processed requests :

   Since there is a reject and only part of the flow enters the system, we use the effective occupation rate here :

   $P_N = P_6 = \frac{(1-\varrho)\varrho^N}{1-\varrho^{N+1}} = 0.07742392,$

$\varrho_e = \frac{\lambda_e}{\mu} = \frac{\lambda(1-P_N)}{\mu} = \frac{5\times(1-0.07742392)}{6} = 0.7688133947$

Duration of activity per day (24 hours) : $24 \times \varrho_e = 24 \times 0.7688133947 = 18,4515214730$ hours $= 1107,0912883839$ minutes $= 66425,4773030345$ seconds.

$Nb = \lfloor \frac{\text{Activity duration}}{\overline{S}} \rfloor = \lfloor \text{Activity duration} \times \mu \rfloor = \lfloor 66425,4773030345 \times 6 \rfloor = \lfloor 398552,8638182071 \rfloor \implies$

$Nb = 398552$ requests **1.0**

4. The average number of waiting requests :

$P_0 = \frac{1-\varrho}{1-\varrho^{N+1}} = 0.231186605289$

$\overline{Q} = \frac{(m\varrho)^m \varrho P_0}{m!(1-\varrho)^2} \left[1 - \varrho^{N-m} - (N-m)(1-\varrho)\varrho^{N-m}\right]$

$\overline{Q} = \frac{0.8333333^2 \times 0.231186605289}{1!(1-0.8333333333)^2} \left[1 - 0.8333333^5 - 5(1-0.8333333)0.8333333^5\right]$

Hence : $\overline{Q} = 1.521349183146$ requests **1.0**

Answer to exercise 03 : (Queueing network : 10 Marks)

Nous avons : $\gamma = 5, m_1 = +\infty, m_2 = 2, m_3 = 1, \mu_1 = 1, \mu_2 = 5, \mu_3 = 10.$

1. Internal routing probability matrix :

$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \alpha & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$ **1.0**, External routing probability matrix : $\begin{pmatrix} 0 \\ 1-\alpha \\ 1-\alpha \end{pmatrix}$ **1.0**

2. Effective arrival rates λ_i :

$\begin{cases} \lambda_1 = \gamma + \frac{1}{3}\lambda_1 + \alpha\lambda_2 + \alpha\lambda_3 \\ \lambda_2 = \frac{1}{3}\lambda_1 \\ \lambda_3 = \frac{1}{3}\lambda_1 \end{cases} \Rightarrow \begin{cases} \frac{2}{3}\lambda_1 = 4 + \frac{2\alpha}{3}\lambda_1 \\ \lambda_2 = \lambda_3 = \frac{1}{3}\lambda_1 \end{cases} \Rightarrow \begin{cases} \frac{2(1-\alpha)}{3}\lambda_1 = 4 \\ \lambda_2 = \lambda_3 = \frac{1}{3}\lambda_1 \end{cases}$

$\Rightarrow \begin{cases} \lambda_1 = \frac{6}{1-\alpha} \\ \lambda_2 = \lambda_3 = \frac{2}{1-\alpha} \end{cases} \Rightarrow \begin{cases} \lambda_1 = \frac{6}{1-\alpha} \\ \lambda_2 = \frac{2}{1-\alpha} \\ \lambda_3 = \frac{2}{1-\alpha} \end{cases}$ **1.5**

Note that $\alpha = 1$ cannot be a solution.

3. Values of α which ensure the stability of the network :

$\begin{cases} FA_1 \text{ Stable} \\ \wedge \\ FA_2 \text{ Stable} \\ \wedge \\ FA_3 \text{ Stable} \end{cases} \Rightarrow \begin{cases} \varrho_1 = 0 < 1 \\ \wedge \\ \varrho_2 = \frac{1}{4(1-\alpha)} < 1 \\ \wedge \\ \varrho_3 = \frac{1}{6(1-\alpha)} < 1 \end{cases} \Rightarrow \begin{cases} 1-\alpha > \frac{1}{4} \\ \wedge \\ 1-\alpha > \frac{1}{6} \end{cases} \Rightarrow \begin{cases} \alpha < \frac{3}{4} \\ \wedge \\ \alpha < \frac{5}{6} \end{cases}$

network stable $\implies \begin{cases} 0 \leq \alpha < \frac{3}{4} \\ \wedge \\ 0 \leq \alpha < \frac{5}{6} \end{cases}$

We conclude that for the network to be stable, α must verify the following condition :

$0 \leq \alpha < \frac{3}{4}$ or $\alpha \in [0, \frac{3}{4}[$ **1.0**

For $\alpha = \frac{1}{2}$:

$\begin{cases} \lambda_1 = 12 \\ \lambda_2 = 4 \\ \lambda_3 = 4 \end{cases} \begin{cases} \varrho_1 = 0 \\ \varrho_2 = \frac{\lambda_2}{m_2\mu_2} = \frac{4}{8} \\ \varrho_3 = \frac{\lambda_3}{m_3\mu_3} = \frac{4}{2\times 6} \end{cases} \Rightarrow \begin{cases} \varrho_1 = 0 \\ \varrho_2 = \frac{1}{2} = 0.5 \\ \varrho_3 = \frac{1}{3} = 0.3333333333 \end{cases}$

Queue 1 : of type $M/M/+\infty$. **Queue 2 :** de type $M/M/2$. **Queue 3 :** de type $M/M/1$.

4. The average number of customers waiting in each queue and in the network :

(a) $\boxed{\overline{Q_1} = 0}$ **0.25**

(b) $\overline{Q_2} = \frac{\varrho_2^2}{1-\varrho_2} = \frac{0.5^2}{1-0.5} \Longrightarrow \boxed{\overline{Q_2} = \frac{1}{2} = 0.5}$ **0.25**

(c) $P_0(3) = \left[\frac{(m_3\rho_3)^{m_3}}{m_3!(1-\rho_3)} + \sum_{k=0}^{m_3-1}\frac{(m_3\rho_3)^k}{k!}\right]^{-1} = \left[\frac{(2\times\frac{1}{3})^2}{2!(1-\frac{1}{3})} + 1 + 2\times\frac{1}{3}\right]^{-1} = \left[\frac{1}{3} + 1 + 2\times\frac{1}{3}\right]^{-1}$

$P_0(3) = \left[\frac{1}{3} + \frac{3}{3} + \frac{2}{3}\right]^{-1} \Rightarrow \boxed{P_0(3) = \frac{1}{2} = 0.5}$ **0.25**

$\zeta_3 = \frac{(m_3\rho_3)^{m_3}}{m_3!(1-\rho_3)}P_0(3) = \frac{(2\times\frac{1}{3})^2}{2!(1-\frac{1}{3})}P_0(3) = \frac{1}{3}\times\frac{1}{2} \Rightarrow \boxed{\zeta_3 = \frac{1}{6} = 0.1666666666}$ **0.25**

$\overline{Q_3} = \frac{\zeta_3\rho_3}{1-\rho_3} = \frac{1\times 3}{6\times 3\times 2} \Rightarrow \boxed{\overline{Q_3} = \frac{1}{12} = 0.0833333333}$ **0.25**

(d) $\overline{Q_R} = \sum_{i=1}^{3}\overline{Q_i} = 0 + \frac{1}{2} + \frac{1}{12} \Rightarrow \boxed{\overline{Q_R} = \frac{7}{12} = 0.5833333333}$ **0.25**

5. The average number of customers in each queue and in the network :

(a) $\overline{N_1} = \overline{R_1} = \frac{\lambda_1}{\mu_1} \Rightarrow \boxed{\overline{N_1} = \frac{12}{5} = 2.4}$ **0.25**

(b) $\overline{N_2} = \overline{Q_2} + \overline{R_2} = 0.5 + 0.5 \Rightarrow \boxed{\overline{N_2} = 1}$ **0.25**

(c) $\overline{N_3} = \overline{Q_3} + \overline{R_3} = \frac{1}{12} + \frac{2}{3} \Rightarrow \boxed{\overline{N_3} = \frac{3}{4} = 0.75}$ **0.25**

(d) $\overline{N_R} = \sum_{i=1}^{3}\overline{N_i} = \frac{12}{5} + 1 + \frac{3}{4} = \frac{48+20+15}{20} \Rightarrow \boxed{\overline{N_R} = \frac{83}{20} = 4.15}$ **0.25**

6. The average residence time in each queue and in the network :

(a) $\overline{T_1} = \frac{1}{\mu_1} = \frac{1}{5} \Rightarrow \boxed{\overline{T_1} = 0.2}$ **0.25**

(b) $\overline{T_2} = \frac{\overline{N_2}}{\lambda_2} = \frac{1}{4} \Rightarrow \boxed{\overline{T_2} = 0.25}$ **0.25**

(c) $\overline{T_3} = \frac{\overline{N_3}}{\lambda_3} = \frac{3}{4\times 4} \Rightarrow \boxed{\overline{T_3} = \frac{3}{16} = 0.1875}$ **0.25**

(d) $\overline{T_R} = \frac{\overline{N_R}}{\lambda_R} = \frac{\overline{N_R}}{\sum_{i=1}^{2}\gamma_i} = \frac{83}{20\times 4} \Rightarrow \boxed{\overline{T_R} = \frac{83}{80} = 1.0375}$ **0.25**

7. The average waiting time in each queue and in the network :

(a) $\boxed{\overline{W_1} = 0}$ **0.25**

(b) $\overline{W_2} = \frac{\overline{Q_2}}{\lambda_2} = \frac{1}{2\times 4} \Rightarrow \boxed{\overline{W_2} = \frac{1}{8} = 0.125}$ **0.25**

(c) $\overline{W_3} = \frac{\overline{Q_3}}{\lambda_3} = \frac{1}{12\times 4} \Rightarrow \boxed{\overline{W_3} = \frac{1}{48} = 0.0208333333}$ **0.25**

(d) $\overline{W_R} = \frac{\overline{Q_R}}{\lambda_R} = \frac{\overline{Q_R}}{\sum_{i=1}^{2}\gamma_i} = \frac{0+\frac{1}{2}+\frac{1}{12}}{4} \Rightarrow \boxed{\overline{W_R} = \frac{7}{48} = 0.1458333333}$ **0.25**

8. The probability that the network is not empty :
Pr(Network not empty) $= 1 - Pr$(Network is empty) $= 1 - (Pr$(Queue$_1$ empty and Queue$_2$ empty and Queue$_3$ empty))
$= (1 - P_0(\text{Queue}_1) \times P_0(\text{Queue}_2) \times P_0(\text{Queue}_3))$
Pr(Network not empty) $= 1 - e^{\frac{-\lambda_1}{\mu_1}} \times P_0(2) \times P_0(3) = 1 - e^{-\frac{12}{5}} \times (1-\rho_2) \times P_0(3)$
$= 1 - 0.0907179532 \times 0.5 \times 0.5 = 1 - 0.0226794883$
Hence : $\boxed{Pr(\text{Network not empty}) = 0.9773205116}$ **1.00**