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**Department of Mathematics and Computer Science** **Academic year: 2024/2025**

**Level: L1 Computer Science**

**Corrected of Analysis 2 Exam**

**<u>Exercise 1</u> (2.5+3+2.5=8 pts)**

1) $\forall n \geq 1$: $u_n = \sum_{k=1}^{n} \frac{n}{n^2+k^2}$. Using the Riemann sums calculate the limit: $\lim_{n\to\infty}(u_n)$.

Answer

$$u_n = \sum_{k=1}^{n} \frac{n}{n^2+k^2}$$
$$= \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+\left(\frac{k}{n}\right)^2} \quad (0.5)$$
$$= \frac{b-a}{n}\sum_{k=1}^{n} f\left(a+\frac{b-a}{n}k\right),$$

where

$$a = 0; b = 1;\ f(x) = \frac{1}{1+x^2}, \quad (2x0.5)$$

so

$$\lim_{x\to\infty}(u_n) = \int_0^1 \frac{1}{1+x^2}dx \quad (0.5)$$
$$= \arctan x|_0^1$$
$$= \frac{\pi}{4} \quad (0.5)$$

2) a) Calculate the integral: $I = \int \frac{3t}{t^2+t-2}dt$. ( Note that: $t^2+t-2 = (t-1)(t+2)$ )

Answer

We put

$$\frac{3t}{t^2+t-2} = \frac{3t}{(t-1)(t+2)}$$
$$= \frac{a}{t+2} + \frac{b}{t-1} \quad (0.5)$$
$$= \frac{(a+b)t+2b-a}{(t-1)(t+2)}. \quad (0.5)$$

So

$$\begin{cases} a+b=3 \\ 2b-a=0 \end{cases}; \quad (0.5)$$

we get

$$\begin{cases} a=2 \\ b=1 \end{cases}. \quad (0.5)$$

So

$$I = \int \frac{3t}{t^2+t-2}dt$$

$$= \int \frac{2}{t+2} + \frac{1}{t-1} dt, \qquad (0.5)$$

we obtain

$$I = 2\ln|t+2| + \ln|t-1| + C \qquad (0.5)$$

b) By integration by change of variable and using the integral $I$, calculate the integral:

$$J = \int \frac{3}{x + \sqrt{x+2}} dx.$$

Answer

(3x0.5)

Putting $t = \sqrt{x+2}$ we get $x = t^2 - 2$ and $dx = 2tdt$, by substitution in the integral $J$ we get

$$J = \int \frac{3}{t^2 - 2 + t} 2tdt$$

$$= 2 \int \frac{3t}{t^2 + t - 2} dt \qquad (0.25)$$

$$= 2I \qquad (0.25)$$

$$= 4\ln|t+2| + 2\ln|t-1| + C. \qquad (0.25)$$

So

$$J = 4\ln\left|\sqrt{x+2} + 2\right| + 2\ln\left|\sqrt{x+2} - 1\right| + C. \qquad (0.25)$$

**<u>Exercise 2</u> (3+1=3 pts)**

1) Find the general solution of the differential equation: $y' - y = x \ldots \ldots (E)$.

Answer

First we calculate the integrating factor:

$$v(x) = e^{\int -1dx} \qquad (0.5)$$

$$= e^{-x}. \qquad (0.25)$$

Multiplying by $v(x)$ gives:

$$e^{-x}.\, y' - e^{-x}.\, y = xe^{-x} \qquad (0.25)$$

or

$$\frac{d}{dx}(e^{-x}y) = xe^{-x}, \qquad (0.25)$$

then

$$e^{-x}y = \int xe^{-x}\, dx, \qquad (0.25)$$

$$= -e^{-x}x + \int e^{-x}\, dx \qquad (0.5)$$

$$= -e^{-x}x - e^{-x} + C \qquad (0.5)$$

and so

$$y = -x - 1 + Ce^{x} \qquad (0.5)$$

2) Deduce the particular solution to equation $(E)$ that achieves $y(0) = 1$.

Answer

We have

$$y(0) = 1 \Leftrightarrow -0 - 1 + Ce^0 = 1$$
$$\Leftrightarrow c = 2.$$ (0.5)

So the particular solution to equation $(E)$ that achieves $y(0) = 1$ is

$$y = -x - 1 + 2e^x.$$ (0.5)

**<u>Exercise 3</u> ( 3+2+1.5+1.5=8 pts)**

1) Find a limited development of order 3 in a neighborhood of 0 for the functions

Answer

$$u(x) = e^x \tan x$$
$$= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right)\left(x + \frac{1}{3}x^3\right)$$ (0.5)
$$= \left(1 + x + \frac{1}{2}x^2\right)(x) + (1)\left(\frac{1}{3}x^3\right)$$ (0.5)
$$= x + x^2 + \frac{5}{6}x^3 + o(x^3).$$ (0.5)

$$v(x) = \frac{x + \frac{1}{6}x^3}{1 - x}$$

$$\begin{array}{c|l} x + \frac{1}{6}x^3 & 1 - x \\ x - x^2 & \overline{x + x^2 + \frac{7}{6}x^3} \\ \hline x^2 + \frac{1}{6}x^3 & \\ x^2 - x^3 & \\ \hline \frac{7}{6}x^3 & \\ \frac{7}{6}x^3 & \\ \hline 0 & \end{array}$$ (3x.5)

So

$$v(x) = x + x^2 + \frac{7}{6}x^3 + o(x^3).$$

2) Using the previous limited developments, calculate the following limit:

$$\lim_{x \to 0}\left(\frac{\frac{\sinh x}{1-x} - e^x \tan x}{x^3}\right).$$

Answer

$$\lim_{x\to 0}\left(\frac{\frac{\sinh x}{1-x}-e^x\tan x}{x^3}\right)=\lim_{x\to 0}\left(\frac{x+x^2+\frac{7}{6}x^3-\left(x+x^2+\frac{5}{6}x^3\right)+o(x^3)}{x^3}\right)$$ (0.5)

$$=\lim_{x\to 0}\left(\frac{\frac{7}{6}x^3-\frac{5}{6}x^3+o(x^3)}{x^3}\right)$$ (0.5)

$$=\lim_{x\to 0}\left(\frac{1}{3}+o(1)\right)$$

$$=\lim_{x\to 0}\left(\frac{1}{3}+\varepsilon(x)\right)$$ (0.5)

$$=\frac{1}{3}.$$ (0.5)

3) a) Find a limited development of order 2 in a neighborhood of $\infty$ for the function

$$h(x)=\ln\left(1+\frac{2}{x}\right).$$

Answer

$$H\left(\frac{1}{t}\right)=\ln(1+2t)$$ (0.5)

$$=2t-\frac{1}{2}(2t)^2+o(t^2)$$ (0.5)

$$=2t-2t^2+o(t^2).$$ (0.25)

Butting $t=\frac{1}{x}$ we get:

$$h(x)=\frac{2}{x}-\frac{2}{x^2}+o\left(\frac{1}{x^2}\right).$$ (0.25)

b) Calculate the limit $\lim_{x\to\infty}(g(x)-2x+2)$ and conclude that the graph $(Cg)$ has an asymptote (denoted $(\Delta)$ ), write an equation for $(\Delta)$.

Answer

$$\lim_{x\to\infty}(g(x)-2x+2)=\lim_{x\to\infty}\left[x^2\left(\frac{2}{x}-\frac{2}{x^2}+o\left(\frac{1}{x^2}\right)\right)-2x+2\right]$$ (0.25)

$$=\lim_{x\to\infty}(2x-2+o(1)-2x+2)$$

$$=\lim_{x\to\infty}\left(o(1)\right)$$ (0.5)

$$=\lim_{x\to 0}\left(\varepsilon(x)\right)$$

$$=0.$$ (0.25)

We conclude that the graph $(Cg)$ has an asymptote $(\Delta)$, and the equation of $(\Delta)$ is:

$$y=2x-2.$$ (2x0.25)