ARBI BEN MHIDI OUM eL BOUAGHI UNIVERSITY
Department of Mathematics and Computer Science
First year Maths **Module Algebra 2,**
Teacher: Soula **2024/2025**

exam in: 14/05/2025

Exercise 1 (5 pts)

Show that $E = \{(x+y+z,\ x-y,\ z)/\ \ x,y,z \in \mathbb{R}^3\}$ is a vector subspace of $\mathbb{R}^3$

Exercise 2 (7.5 pts)

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x,y) = (x-y, -3x+3y)$$

1) Show that f is a linear application.
2) Give a basis of $Ker(f)$ and a basis of $Im(f)$.
3) Show that f is neither injective no surjective.

Exercise 3 (7.5 pts)

Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear application whose matrix in the canonical basis
of $\mathbb{R}^4$ and $\mathbb{R}^3$ is

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & -2 & 5 & -11 \end{pmatrix}$$

1) Determine a basis of the $Ker(f\ )$.
2) Determine a basis of the $Im(f)$. What is the rank of f ?

جامعة العربي بن مهيدي أم البواقي

قسم الرياضيات والاعلام الالي

السنة الاولى رياضيات **السنة الجامعية : 2025/2024**

المادة : الجبر 2 **الاستاذة : ي. صولة**

الامتحان: يوم 2025/05/14

<u>التمرين 1</u> : (5 نقاط)

برهن أن $E = \{ (x+y+z,\ x-y,\ z)\ /\ x,y,z \in \mathbb{R}^3 \}$ هي فضاء شعاعي جزئي لـ$\mathbb{R}^3$

<u>التمرين 2</u> : (7.5 نقاط)

ليكن $f: \mathbb{R}^2 \to \mathbb{R}^2$ مُعرّفًا كمايلي

$$f(x,y) = (x-y,\ \ -3x+3y)$$

1) بيّن أن f تطبيق خطي
2) أعطِ أساس $Ker(f)$ وأساس $Im(f)$
3) بيّن أن f ليس متباين ولا غامر.

<u>التمرين 3</u> : (7.5 نقاط)

ليكن $f: \mathbb{R}^4 \to \mathbb{R}^3$ التطبيق الخطي الذي تكون مصفوفته في الأساس القانوني لـ $\mathbb{R}^4$ و $\mathbb{R}^3$ هي

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & -2 & 5 & -11 \end{pmatrix}$$

1) حدد أساس $Ker(f)$
2) حدد أساس $Im(f)$
3) ما رتبة التطبيق f ؟

بالتوفيق

ARBI BEN MHIDI OUM eL BOUAGHI UNIVERSITY
Department of Mathematics and Computer Science
First year Maths **Module Algebra 2,**
Teacher: Soula **2024/2025**

Typical answer

Exercise 1(5 pts)

• $0_{\mathbb{R}^3} = (0,0,0) \in E$ because $(0,0,0) = (0+0+0, 0-0, 0))$
So $E \neq \phi$ ....**(2 pts)**

• $\forall X, Y \in E,\ \lambda,\ \mu \in \mathbb{R}$ show that $\lambda X + \mu Y \overset{?}{\in} E$; on a :
$X \in E, \exists(x,y,z) \in \mathbb{R}^3 / X = (x+y+z, x-y, z)$,
$Y \in E, \exists(x', y', z') \in \mathbb{R}^3 / X = (x'+y'+z', x'-y', z')$,
$\lambda X + \mu Y = \lambda(x+y+z, x-y, z) + \mu(x'+y'+z', x'-y', z')$
$= (\lambda x + \lambda y + \lambda z, \lambda x - \lambda y, \lambda z) + (\mu x' + \mu y' + \mu z', \mu x' - \mu y', \mu z')$
$= ((\lambda x + \mu x') + (\lambda y + \mu y') + (\lambda z + \mu z'), (\lambda x + \mu x') - (\lambda y + \mu y'), (\lambda z + \mu z'))$
from where $\exists x" = \lambda x + \mu x' \in \mathbb{R}$, $\exists y" = \lambda y + \mu y' \in \mathbb{R}$, and $\exists z" = \lambda z + \mu z' \in \mathbb{R}$
So, $\lambda X + \mu Y \in E$ ....**(3 pts)**
Then E is a vector subspace of $\mathbb{R}^3$

Exercise 2 (7.5 pts)

1) Let $(x,y), (x',y') \in \mathbb{R}^2$, $\forall \lambda,\ \mu \in \mathbb{R}$,

$$
\begin{aligned}
f(\lambda(x,y) + \mu(x',y')) &= f(\lambda x + \mu x', \lambda y + \mu y') \\
&= (\lambda x + \mu x' - \lambda y - \mu y', -3(\lambda x + \mu x') + 3(\lambda y + \mu y')) \\
&= (\lambda(x-y) + \mu(x'-y'), \lambda(-3x+3y) + \mu(-3x'+3y')) \\
&= (\lambda(x-y), \lambda(-3x+3y)) + (\mu(x'-y'), \mu(-3x'+3y')) \\
&= \lambda((x-y), (-3x+3y)) + \mu((x'-y'), (-3x'+3y')) \\
&= \lambda f((x,y) + \mu f(x',y')
\end{aligned}
$$

So, f is a linear application..**(2 pts)**

$$(x,y) \in ker(f) \Leftrightarrow (x-y, -3x+3y) = (0,0) \Leftrightarrow \begin{cases} 0 = x - y \\ 0 = -3x + 3y \\ \Leftrightarrow x = y \end{cases}$$

So $(x,x) = x(1,1)$, $(1,1)$ is a non-zero vector
which generates $ker(f)$, it is a basis of $ker(f\ )$.........................**(1.25 pts)**
$f(e_1) = (1-0, -3\times 1 + 3\times 0) = (1,-3) = e_1 - 3e_2$ and
$f(e_2) = ((0-1, -3\times 0 + 3\times 1) = (-1,3) = -e_1 + 3e_2$
$Im(f) = Vect(f(e_1), f(e_2)) = Vect(e_1 - 3e_2, -e_1 + 3e_2) = Vect(e_1 - 3e_2)$
$e_1 - 3e_2$ is a non-zero vector that generates $Im(f)$,
it is a base of $Im(f)$...**(1.25 pts)**

3) $f(1,1) = (0,0) = f(0,0)$ and yet $(1,1) \neq (0,0)$
so f is not injective...**(1.5 pt)**
We will show that $(1,0)$ has no antecedent. Suppose there exists (x,y)
such that $(1,0) = f(x,y) \Leftrightarrow (1,0) = (x-y, -3x+3y)$

$$\Leftrightarrow \begin{cases} 1 = x - y \\ 0 = -3x + 3y \end{cases}$$

$$\Leftrightarrow \begin{cases} 1 = x - y \\ x = y \end{cases}$$

$$\Leftrightarrow \begin{cases} 1 = 0 \\ x = y \end{cases} \text{ it's impossible.}$$

so f is not surjective..**(1.5 pt)**
Or, f is an endomorphism so f is injective if and only if f is surjective.
Here, f is not injective so f is not surjective.

<u>Exercise 3</u> (7.5 pts)

1) Let $X \in \mathbb{R}^4$, So $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

$$X \in Ker(f) \Leftrightarrow AX = O_{\mathbb{R}^3} \Leftrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & -2 & 5 & -11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 + x_3 + 3x_4 = 0 \\ x_1 + x_2 + 2x_3 + x_4 = 0 \\ x_1 - 2x_2 + 5x_3 - 11x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -3x_3 \\ x_2 = x_3 \\ x_4 = 0 \end{cases}$$

$X = (-3x_3, x_3, x_3, 0) = x_3(-3, 1, 1, 0)$
Then $(-3, 1, 1, 0)$ is a basis of $ker(f)$, so $dim\ ker(f) = 1$...........**(3 pts)**

2) According to the rank theorem
$dim\ ker(f) + dim\ \mathrm{Im}(f) = dim\ \mathbb{R}^4 = 4$
So, $dim\ \mathrm{Im}(f) = 3 = .dim\ \mathbb{R}^3$...**(3 pts)**
Then $\mathrm{Im}(f) = \mathbb{R}^3$ and $rg(A) = dim\,\mathrm{Im}(f) = 3$......................**(1.5 pts)**