L'arbi Ben M'hidi University

Faculty: Exact sciences , natural and life sciences
Department: MI
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Module Algebra 1

Correction of Exam n^0 1

Exercice 1:(7 **pts**)
1) Prove that $\overline{P \Rightarrow Q} \Leftrightarrow P \wedge \overline{Q}$.

P	Q	$P \Rightarrow Q$	$\overline{P \Rightarrow Q}$	$\overline{Q}$	$P \wedge \overline{Q}$
1	1	1	0	0	0
0	0	1	0	1	0
1	0	0	1	1	1
0	1	1	0	0	0

(4 pts)

(1 **pts for each column)**
2) Show by recurrence that

$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(a) For $n = 1$ we have : $1^2 = \frac{1(2)(3)}{6}$, $P(1)$ is true.......... (0.5 **pts**)
(b) - Suppose that

$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

is true and we will prove that

$$1^2 + 2^2 + ... + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6} \text{.......... (1 \textbf{pts})}$$

We have

$$\begin{aligned} 1^2 + 2^2 + ... + (n+1)^2 &= \left(1^2 + 2^2 + ... + +n^2\right) + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \text{...................... (1.5 \textbf{pts})} \end{aligned}$$

Exercice 2 : (6 **pts**)

1. Let the relation $\boldsymbol{R}$ defined by :

$$(x,y)\ \boldsymbol{R}\ (x',y') \Leftrightarrow |x-x'| \leq y'-y.$$

Prove that $\boldsymbol{R}$ is an order relation.

i) We will prove that $\boldsymbol{R}$ is reflexive i.e.

$$\forall (x,y) \in \mathbb{R}^2 : (x,y)\, \boldsymbol{R}\, (x,y) \text{ .... } (0.\mathbf{5\ pts})$$

we have $\forall (x,y) \in \mathbb{R}^2 : |x-x| \leq y-y$.. So $(x,y)\, \boldsymbol{R}\, (x,y)$ **..............**(1.0 **pts**)

ii) We will prove that $\boldsymbol{R}$ is antisymmetric i.e.

$$\forall (x,y), (x',y') \in \mathbb{R}^2 : (x,y)\, \boldsymbol{R}\, (x',y') \wedge (x',y')\, \boldsymbol{R}\, (x,y) \Rightarrow (x,y) = (x',y') \text{ .... } (0.\mathbf{5\ pts})$$

we have

$$\begin{aligned}
(x,y)\, \boldsymbol{R}\, (x',y') \wedge (x',y')\, \boldsymbol{R}\, (x,y) \quad &\Rightarrow \quad \left\{ \begin{array}{l} |x-x'| \leq y'-y \\ |x'-x| \leq y-y' \end{array} \right. \\
&\Rightarrow \quad 2\,|x-x'| = 0 \\
&\Rightarrow \quad x = x' \\
&\Rightarrow \quad y'-y \leq 0 \wedge y-y' \geq 0 \\
&\Longrightarrow \quad y = y' \\
&\Rightarrow \quad (x,y) = (x',y')
\end{aligned}$$

$\Rightarrow \boldsymbol{R}$ is antisymmetric**..............**(1.0 **pts**)

iii) We will prove that $\boldsymbol{R}$ is transitive i.e.

$$\forall (x,y), (x',y'), (x",y") \in \mathbb{R}^2 : (x,y)\, \boldsymbol{R}\, (x',y') \wedge (x',y')\, \boldsymbol{R}\, (x",y") \Rightarrow (x,y)\, \boldsymbol{R}\, (x",y") \text{ .... } (0.\mathbf{5\ pts})$$

we have $(x,y)\ \boldsymbol{R}\ (x',y') \Rightarrow |x-x'| \leq y'-y$

and

$(x',y')\ \boldsymbol{R}\ (x",y") \Rightarrow \left|x'-x"\right| \leq y-y"$

$\Rightarrow y-y' \leq x-x' \leq y'-y$ and $y"-y \leq x'-x" \leq y-y"$

$\Rightarrow \left|x-x"\right| \leq y"-y$

so $(x,y)\, \boldsymbol{R}\, (x",y")$ and $\boldsymbol{R}$ is transitive**..............**(1.0 **pts**)

therefore $\boldsymbol{R}$ is an order relation.

2. This ordrer is partial. (take a counter example)**.............**(1.5 **pts**)

Exercice 3:(7 **pts**)

$$x * y = \frac{x+y}{1+xy}$$

i) Prove that $(G,*)$ is a group.

1. Neutral element (1.5 **pts**)

Let e the neutral element of G,so

$\forall x \in G\ : x * e = e * x = x$........... (0.**5 pts**)

$\Rightarrow x * e = \dfrac{x+e}{1+xe} = x..$

$\Rightarrow e = 0$ is the neutral element of G............. (1 **pts**)

2. Symetrical element(1.**5 pts**)

Let x' the symetrical element of x, so we have

$x * x' = x' * x = 0$............ (0.**5 pts**)

$\Rightarrow \dfrac{x+x'}{1+xx'} = 0.$

$\Rightarrow x' = -x$ is the symetrical element of x............. (1 **pts**)

3. Associativity (3.0 **pts**)

Let $x,\ y,\ z \in G$, prove that

$[x * y] * z = x * [y * z]$ .... (0.**5 pts**)

we have

$[x * y] * z = \dfrac{x+y}{1+xy} * z = \dfrac{x+y+z+xyz}{1+xy+xz+yz}$....... (1)

$x * [y * z]\,. = x * \left[\dfrac{y+z}{1+yz}\right].$

$= \dfrac{x+y+z+xyz}{1+xy+xz+yz}$........ (2)

So (1) = (2) and $*$ is associative............. (2.5**pts**)

we conclude that $(G = \mathbb{R}^+\ , *)$ is a group.

3. commutativity (1.0 **pts**)

we have

$$x * y = \frac{x+y}{1+xy} = y * x$$

So $*$ is commutative.

Bonne chance.

Pr.Rezzag.S