L'arbi Ben M'hidi University

Faculty: Exact Sciences, Natural and Life sciences **In:** 09/01/2025
Department : Mathematics and Computer Science. **Duration:** 1h30min
Level: Second year of Bachelor Mathematics

Mathematical logic exam

Exercise 1: (7 pts)
Let's consider a propositional langage where
$A =$"Amel comes to the party",
$B =$"Bahia comes to the party",
$C =$"Camilia comes to the party",
$D =$"Dalila comes to the party".
Formalize the following sentences:
1. "If Dalila comes to the party then Bahia and Camilia come too"
2. "Camilia comes to the party only if Amel and Bahia do not come"
3. "Dalila comes to the party if and only if Camilia comes and Amel doesn't come"
4. "If Dalila comes to the party, then, if Camilia doesn't come then Amel comes"
5. "Camilia comes to the party with the condition that Dalila doesn't come, but, if Dalila comes, then Bahia doesn't come"
6. "A necessary condition for Amel coming to the party, is that, if Bahia and Camilia aren't coming, then Dalila comes"
7. "Amel, Bahia and Camilia come to the party if and only if Dalila doesn't come, but, if neither Amel nor Bahia come, then Dalila comes only if Camilia comes"

Exercise 2: (5 pts)
1/ Use the truth tables method to determine whether the following formula

$$(P \vee Q) \wedge (P \Longrightarrow (R \wedge Q)) \wedge (Q \Longrightarrow (\rceil R \wedge P))$$

is an antilogy.
2/ Let α be an antilogy and β any formula. Show that $\alpha \Longrightarrow \beta$ is a tautology.

Exercise 3: (5 pts)

The Fibonacci sequence is given by $\begin{cases} F_0 = 0 \\ F_1 = 1 \\ \forall n \in \mathbb{N} : F_{n+2} = F_{n+1} + F_n \end{cases}$

Let $\varphi = \frac{1+\sqrt{5}}{2}$ and $\varphi' = \frac{1-\sqrt{5}}{2}$ (φ is called the golden ratio). We have φ and φ' are solutions of the equation $x^2 - x - 1 = 0$
Question: By absurdity and well ordering proof show that for all $n \geq 1$ we have $F_n \leq \varphi^{n-1}$.

Exercise 4: (3 pts)

With contrapositive show that: for $x, y \in \mathbb{Z}$; If $5 \nmid xy$ then $5 \nmid x$ and $5 \nmid y$.

Correction

EX01:

1/ $D \Longrightarrow B \wedge C$.... (1)

2/ $C \Longrightarrow \overline{A} \wedge \overline{B}$..... (1)

3/ $D \Longleftrightarrow C \wedge \overline{A}$.... (1)

4/ $D \Longrightarrow (\overline{C} \Longrightarrow A)$ ... (1)

5/ $(\overline{D} \Longrightarrow C) \wedge (D \Longrightarrow \overline{B})$ ... (1)

6/ $A \Longrightarrow (\overline{B} \wedge \overline{C}) \Longrightarrow D$... (1)

7/ $[(A \wedge B \wedge C) \Longleftrightarrow \overline{D}] \wedge (\overline{A \wedge B}) \Longrightarrow (D \Longrightarrow C)$ .... (1)

EX02:

1/ $(P \vee Q) \wedge (P \Longrightarrow (R \wedge Q)) \wedge (Q \Longrightarrow (\rceil R \wedge P))$ ... $(*)$ ... (3.5)

P	Q	R	$\rceil R$	$P \vee Q$	$R \wedge Q$	$P \Longrightarrow (R \wedge Q)$	$\rceil R \wedge P$	$Q \Longrightarrow (\rceil R \wedge P)$	$(*)$
1	1	1	0	1	1	1	0	0	0
1	1	0	1	1	0	0	1	1	0
1	0	1	0	1	0	0	0	1	0
1	0	0	1	1	0	0	1	1	0
0	1	1	0	1	1	1	0	0	0
0	1	0	1	1	0	1	0	0	0
0	0	1	0	0	0	1	0	1	0
0	0	0	1	0	0	1	0	1	0

This formula is an antilogy (0.25)

2/

α	β	$\alpha \Longrightarrow \beta$
0	1	1
0	0	1

So in two cases $\alpha \Longrightarrow \beta$ is true, then $\alpha \Longrightarrow \beta$ is a tautology.....(1.25)

EX03:

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ \forall n \in \mathbb{N} : F_{n+2} = F_{n+1} + F_n ..... (*) \end{cases}$$

By absurdity and well ordering proof showing that $\forall n \geq 1 : F_n \leq \varphi^{n-1}$.

Assume that it exits a set $C \neq \emptyset$, such that $C = \{n \in \mathbb{N}, F_n > \varphi^{n-1}\}$ (0.5)

We have $C \subset \mathbb{N}$, by well ordering principle $\exists n_0 \in C, n_0$ is the minimum of C.(it means that $\forall n \in C : n_0 \leq n$) (0.5)

Then $F_{n_0} > \varphi^{n_0-1}$, in other part we have $F_1 \leq \varphi^0 = 1$... (0.5) , so $1 \notin C$ and $n_0 - 1 \notin C$ as a result $F_{n_0-1} \leq \varphi^{n_0-2}$.... (0.5)

And $F_0 = 0 \leq \varphi^{-1}$ is also true(0.5), so $F_{n_0-2} \leq \varphi^{n_0-3}$.... (0.5)

By $(*)$ we get $F_{n_0} = F_{n_0-1} + F_{n_0-2}$.... (0.5)

$\Longrightarrow F_{n_0} \leq \varphi^{n_0-2} + \varphi^{n_0-3}$... (0.25)

$\Longrightarrow F_{n_0} \leq \varphi^{n_0-3}(\varphi + 1)$... (0.25) $[**]$

But φ is a solution for the equation $x^2 - x - 1 = 0$ so $\varphi^2 - \varphi - 1 = 0 \Longrightarrow \varphi^2 = \varphi + 1$.. (0.5) . $[***]$

Remplace $[***]$ in $[**]$ we get $F_{n_0} \leq \varphi^{n_0-3}\varphi^2$ then $F_{n_0} \leq \varphi^{n_0-1}$... (0.5)

we have a contradiction then $C = \emptyset$ and $n \geq 1$ we have $F_n \leq \varphi^{n-1}$...(0.5)

EX04:

$\forall x, y \in \mathbb{Z}; : 5 \nmid xy \Longrightarrow 5 \nmid x \ \wedge \ 5 \nmid y$.

The contrapositive is $\forall x, y \in \mathbb{Z};$:$5/x \ \vee \ 5/y. \Longrightarrow 5/xy$ ....(1)

a/ Assume that $5/x$ so $\exists k \in \mathbb{Z}$ such that $x = 5k$ then $xy = 5ky = 5k'$ where $k' = ky \in \mathbb{Z}$ as a result $5/xy$.......... (1)

b/ Assume now that $5/y$ so $\exists l \in \mathbb{Z}$ such that $y = 5l$ then $xy = 5lx = 5l'$ where $l' = lx \in \mathbb{Z}$ as a result $5/xy$.......... (1)