Oum El bouaghi University
MI Department
S1 -Maths 03-
Exam (Optimization)

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**Exercise 1** ***Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous and coercive function. Prove that $f$ attains its minimum, i.e., there exists a point***

$$\mathbf{x}^* \in \mathbb{R}^n, \textit{ such that } \mathbf{f}(\mathbf{x}^*) = \min \mathbf{f}(\mathbf{x})$$

***- Does the theorem hold if $f$ is coercive but not continuous? Justify your answer***

***- Do these conditions guarantee the existence of both a minimum and a maximum for $f$?***

***If not provide the theorem that guarantees the existence of both a minimum and a maximum (without proof)***

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**Exercise 2** ***Let $f : \mathbb{R}^n \to \mathbb{R}$***

***Suppose that at a point $x^*$, we have $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is negative definite***

***Prove that $x^*$ is a strict local maximum of $f$***

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**Exercise 3** ***Consider the function***

$$f(x, y) = y^2 + xy\ln(x).$$

***- Find the critical point***

***- Give the nature***

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**Exercise 4** ***Consider the following function***

$$f(x, y) = x^2 + y^2 - 2x - 2y$$

***- What is the general form of the Gradient Descent algorithm?***

***- Does the Gradient Descent algorithm help in finding a local or global minimum?***

***- What happens if the step size is too large?***

***- What happens if the step size is too small?***

***-Start with the initial point $(x_0, y_0) = (2, 3)$ and $\alpha = 0.1$. Apply the gradient descent to find the value of iteration $(x_2, y_2)$.***

# Correction Exam (Optimization)

**Exercise 01**(6 *pts*)

1- The coercivity implies that the function grows to infinity as x moves away from the origin,

which ensures that the function cannot have arbitrarily low values at infinity.

To apply the Weierstrass theorem, we need to show that $f$ attains a minimum on a compact set.

Since $f$ is coercive, we know that values of $f$ will become arbitrarily large outside of a sufficiently large ball, say **B(0,R)**

Thus, the minimization problem can be restricted to a compact set $\overline{\mathbf{B}(\mathbf{0},\mathbf{R})}$ On this compact set, the function f is continuous,

and by the Weierstrass extreme value theorem, a continuous function on a compact set attains its minimum. *( 03 points)*

2- If $f$ is coercive but not continuous, the theorem may not hold, because continuity is crucial to guarantee the existence of an actual minimum.*( 01 points)*

3- The given conditions (coercivity and continuity) do not guarantee the existence of both a minimum and a maximum*( 01 points)*

4- Theorem that guarantees both minimum and maximum (without proof):The Weierstrass extreme value theorem *( 1 points)*

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**Exercise 02**(04 *pts*)

**See (The sufficient conditions for optimality in the course)***( 04 points)*

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**Exercise 03** (05 *pts*)

Given function:

$$f(x,y) = y^2 + xy\ln(x).$$

**- The critical point (**Critical points occur where the first-order partial derivatives are equal to zero)

Critical points: $(1,0)$ and $\left(\frac{1}{e}, \frac{1}{2e}\right)$*( 02 points)*

**-** Determine the nature of the critical points (To determine the nature, we use the second derivative test by computing the Hessian matrix.)

At $(1,0)$: $H_f = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$Since $D = -1 < 0$, the point $(1,0)$ is a saddle point.*( 01.5 points)*

At $\left(\frac{1}{e}, \frac{1}{2e}\right)$: $H_f = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$Since $D > 0$, and $a_{11} > 0$ , the point $\left(\frac{1}{e}, \frac{1}{2e}\right)$ is a local minimum point.*( 01.5 points)*

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**Exercise 04**(5 *pts*)

**Consider the following function**

$$f(x,y) = x^2 + y^2 - 2x - 2y$$

**-**General Form of the Gradient Descent Algorithm*( 01 points)*

$$\begin{cases} (x_0, y_0) \text{ given} \\ (x_{k+1}, y_{k+1}) = (x_k, y_k) - \alpha \nabla f(x_k, y_k) \end{cases}$$

**- Does the Gradient Descent algorithm help in finding a local or global minimum?***( 0.5 points)*

For convex functions, Gradient Descent converges to the global minimum,

For non-convex functions, gradient descent may get trapped in local minima.

**- What happens if the step size is too large?***( 0.5 points)*

If the step size $\alpha$ is too large, the algorithm may: Overshoot the minimum and fail to converge,

**- What happens if the step size is too small?***( 0.5 points)*

If the step size $\alpha$ is too small, the algorithm may: Converge very slowly, taking many iterations to reach the minimum,

**-Start with the initial point** $(x_0, y_0) = (2, 3)$ **and** $\alpha = 0.1$. **Apply the gradient descent to find the value of iteration** $(x_2, y_2)$**.***( 02.5 points)*

First iteration (k = 1) $(x_1, y_1) = (1.8, 2.6)$ and $\|\nabla f\| = 3.58$

Second iteration (k = 2) $(x_2, y_2) = (1.64, 2.28)$

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