Exo 1 :

① $H_1 = \frac{P_x^2}{2m} + V(x) = \boxed{\frac{P_x^2}{2m} + x^\lambda}$ (une seule particule)

1 point

pour N particules

$$H = \boxed{\sum_{i=1}^{N} \left(\frac{P_{x_i}^2}{2m} + x_i^\lambda \right)}$$

1 point

② $Z_1 = \frac{1}{h} \int dx \, dP_x \, e^{-\beta H_1} = \frac{1}{h} \int dx \, dP_x \, e^{-\beta \left(\frac{P_x^2}{2m} + x^\lambda \right)}$

$$= \frac{1}{h} \int_{-\infty}^{\infty} e^{-\frac{\beta P_x^2}{2m}} dP_x \cdot \int_0^{\infty} dx \, e^{-\beta x^\lambda}$$

1 point + 2 points

$$= \frac{1}{h} \sqrt{\frac{2m\pi}{\beta}} \cdot \frac{1}{\lambda \beta^{1/\lambda}} \Gamma\left(\frac{1}{\lambda}\right)$$

$$= \boxed{\frac{1}{h} \sqrt{2m\pi k_B T} \cdot \frac{1}{\lambda} (k_B T)^{1/\lambda} \Gamma\left(\frac{1}{\lambda}\right)}$$

$$\Rightarrow \boxed{F(T) = \frac{1}{h} \sqrt{2m\pi k_B T}}$$

1 point

③ $Z(N) = \frac{1}{N!} \frac{1}{h^N} (2m\pi k_B T)^{N/2} \frac{1}{\lambda^N} (k_B T)^{N/\lambda} \left(\Gamma\left(\frac{1}{\lambda}\right)\right)^N$

$$= \frac{1}{N!} C \cdot (k_B T)^{\frac{N}{2} + \frac{N}{\lambda}} = \boxed{\frac{1}{N!} C^N \cdot (k_B T)^{\left(\frac{\lambda+2}{2\lambda}\right) N}}$$

avec $C^N = \frac{1}{h^N} \cdot (2m\pi)^{N/2} \cdot \frac{1}{\lambda^N} \cdot \left(\Gamma\left(\frac{1}{\lambda}\right)\right)^N = cte$

2 points

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4) $F = -\frac{1}{\beta}\log Z = -k_B T \log\left[\frac{1}{N!} C^N \cdot (k_B T)^{\left(\frac{d+2}{2d}\right)N}\right]$

$= -k_B T \left\{ -\log N! + \log C^N + N\left(\frac{d+2}{2d}\right)\log(k_B T)\right\}$

on utilise la formule de Stirling

$\log N! \simeq N \log N - N$

$F = -N k_B T \left\{ 1 - \log N + \log C + \left(\frac{d+2}{2d}\right)\log(k_B T)\right\}$

$= -N k_B T \left\{ 1 + \log \frac{C (k_B T)^{\frac{d+2}{2d}}}{N} \right\}$ 2pont

5) $S = -\frac{\partial F}{\partial T} = N k_B \left\{ 1 + \log \frac{C}{N}(k_B T)^{\frac{d+2}{2d}} \right\}$

$+ N k_B T \left\{ \frac{\cancel{\frac{C}{N}} (k_B T)^{\frac{d+2}{2d}-1} \cdot (k_B) \cdot \frac{d+2}{2d}}{\cancel{\frac{C}{N}} (k_B T)^{\frac{d+2}{2d}}} \right\}$

$= N k_B \left\{ 1 + \log \frac{C}{N}(k_B T)^{\frac{d+2}{2d}} \right\} + \left(\frac{d+2}{2d}\right) N k_B$

$= \boxed{N k_B \left\{ 1 + \frac{d+2}{2d} + \log \frac{C}{N}(k_B T)^{\frac{d+2}{2d}} \right\}}$ 2pont

6) $U = -\frac{\partial \log Z}{\partial \beta}$.

on sait : $\log Z(N) = N \left\{ 1 + \log \frac{C}{N}\left(\frac{1}{\beta}\right)^{\frac{d+2}{2d}} \right\}$

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$$U = -N \frac{\partial}{\partial \beta} \left\{ \log \frac{C}{N} \left(\frac{1}{\beta}\right)^{\frac{d+2}{2d}} \right\}$$

$$= -N \frac{\frac{C}{N} \cdot \left(\frac{1}{\beta}\right)^{\frac{d+2}{2d}-1} \left(-\frac{1}{\beta^2}\right)}{\frac{C}{N}\left(\frac{1}{\beta}\right)^{\frac{d+2}{2d}}} \cdot \left(\frac{d+2}{2d}\right)$$

$$= \frac{N}{\beta} \cdot \left(\frac{d+2}{2d}\right) = \boxed{N K_B T \cdot \left(\frac{d+2}{2d}\right)}$$ 2 point

(7) $C_V = \frac{\partial U}{\partial T} = N K_B \left(\frac{d+2}{2d}\right)$. 1 point

(8) le cas de l'OH : $d = 2$ 1 point

Exo 2 : H pour N oscillateurs.

a) $H = \sum_i \vec{P}_i^2 + Q_i^2$ 1 point

(b) $\omega = \int_{H \leq E} d^{3N}P \cdot d^{3N}Q = \int_{\sum_i^N \vec{P}_i^2 + Q_i^2 \leq E} dp_1 \dots dp_{3N} \cdot dq_1 \dots dq_{3N}$

un chgt $z_1 \dots z_{6N}$

$$= \int_{\sum_{i=1}^{6N} z_i^2 \leq (\sqrt{E})^2} dz_1 \dots dz_{6N} = \frac{\pi^{\frac{6N}{2}} (E^{1/2})^{6N}}{\Gamma\left(\frac{6N}{2}+1\right)}$$

$$= \boxed{\frac{\pi^{3N} \cdot E^{3N}}{\Gamma(3N+1)}}$$ 3 point

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$h = 1$

$$\Omega = \frac{1}{h^{3N}} \frac{\partial \omega}{\partial E} \Big/$$

$$\Rightarrow \Omega = \frac{\partial \omega}{\partial E} = \frac{3N \pi^{3N} E^{3N-1}}{\Gamma(3N+1)}$$

$$= \boxed{\frac{\pi^{3N} E^{3N-1}}{\Gamma(3N)}}$$

2points