Correction of Mathematics 1 Exam

Exercise 1

5pts Given:

$$S_n = \sum_{k=1}^{n} k(k+1) \tag{1}$$

$$T_n = \sum_{k=1}^{n} k^2 \tag{2}$$

1. Compute S_1 and T_2

$$S_1 = 1(1+1) = 2$$
$$T_2 = 1^2 + 2^2 = 1 + 4 = 5$$

2. Proof by Induction for $S_n = \frac{n(n+1)(n+2)}{3}$

Base case: For $n = 1$,

$$S_1 = 2 = \frac{1(2)(3)}{3}$$

which holds.

Inductive step: Assume it holds for n,

$$S_n = \frac{n(n+1)(n+2)}{3}$$

For $n + 1$,

$$\begin{aligned} S_{n+1} &= S_n + (n+1)(n+2) \\ &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \end{aligned}$$

Factoring $(n+1)(n+2)$:

$$S_{n+1} = \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3}$$

$$S_{n+1} = \frac{(n+1)(n+2)(n+3)}{3}$$

Thus, the formula holds by induction.

3. Compute T_n

We use:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{3}$$

$$T_n = S_n - \sum_{k=1}^{n} k \tag{4}$$

$$T_n = \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} \tag{5}$$

Common denominator (6):

$$T_n = \frac{2n(n+1)(n+2) - 3n(n+1)}{6}$$

$$T_n = \frac{n(n+1)(2n+1)}{6}$$

which is the standard formula for the sum of squares.

Exercise 2

4pts Given the relation:

$$(x, y)R(x', y') \iff x(1+y') = x'(1+y) \tag{6}$$

1. Prove R is an equivalence relation

Reflexivity:

$$x(1+y) = x(1+y) \quad \forall (x, y)$$

so R is reflexive.

Symmetry: Assume $(x,y)R(x',y')$, i.e.,

$$x(1+y') = x'(1+y)$$

which is symmetric, so R is symmetric.

Transitivity: If $(x,y)R(x',y')$ and $(x',y')R(x'',y'')$, then

$$x(1+y') = x'(1+y), \quad x'(1+y'') = x''(1+y')$$

Multiplying both,

$$x(1+y'') = x''(1+y)$$

Thus, R is transitive and therefore an equivalence relation.

2. Equivalence Class of $(-1, \frac{1}{2})$

Solve:

$$-1(1+y') = x'(1+\frac{1}{2})$$

$$-1-y' = x' \cdot \frac{3}{2}$$

$$x' = -\frac{2(1+y')}{3}$$

which describes the equivalence class.

Exercise 3

5pts

Function

$$f(x) = \frac{1}{1-e^{2x-1}}.$$

Domain: Solve

$$1-e^{2x-1} \neq 0.$$

That is,

$$e^{2x-1} \neq 1 \quad \Longrightarrow \quad 2x-1 \neq 0 \quad \Longrightarrow \quad x \neq \frac{1}{2}.$$

So,

$$D_f = \mathbb{R} \setminus \left\{\frac{1}{2}\right\}.$$

Compute $f(3)$ and $f\left(-\frac{1}{2}\right)$:

$$f(3) = \frac{1}{1-e^{6-1}} = \frac{1}{1-e^{5}},$$

$$f\left(-\frac{1}{2}\right) = \frac{1}{1-e^{-2}}.$$

Injectivity:
Assume $f(x) = f(y)$, meaning:

$$\frac{1}{1-e^{2x-1}} = \frac{1}{1-e^{2y-1}}.$$

This implies

$$1-e^{2x-1} = 1-e^{2y-1},$$

so that

$$e^{2x-1} = e^{2y-1}.$$

Taking the natural logarithm on both sides yields:

$$2x-1 = 2y-1,$$

and thus,

$$x = y.$$

Therefore, f is injective.

Surjectivity: Since $f(x)$ does not take all real values, it is not surjective.

Function $g(x)$: Study Continuity.

Check at $x = 0$:

$$\lim_{x\to 0^-} g(x) = 2(0) - 4 = -4,$$

$$\lim_{x\to 0^+} g(x) = \frac{1}{2}(0) - 2 = -2.$$

Since $-4 \neq -2$, $g(x)$ is discontinuous at $x = 0$.

Check at $x = 4$:

$$\lim_{x\to 4^-} g(x) = \frac{1}{2}(4) - 2 = 0,$$

$$\lim_{x\to 4^+} g(x) = 2(16) - 5(4) + 4 = 32 - 20 + 4 = 16.$$

Since $0 \neq 16$, $g(x)$ is discontinuous at $x = 4$.

Exercise 4

6pts We are given the set
$$A = \mathbb{R} \times \mathbb{R},$$
with the operations defined for all $(x, y), (x', y') \in A$ by
$$(x, y) + (x', y') = (x + x',\ y + y')$$
and
$$(x, y) \times (x', y') = \big(xx',\ x\,y' + x'\,y\big).$$
We now detail the steps to prove that $(A, +, \times)$ is a commutative ring.

1. Proof that $(A, +)$ is a Commutative Group

To show that $(A, +)$ is a commutative (abelian) group, we verify the following properties:

Closure

For any $(x, y), (x', y') \in A$, we have
$$(x, y) + (x', y') = (x + x',\ y + y').$$
Since the sum of real numbers is real, $(x + x',\ y + y') \in A$.

Associativity

Let $(x, y), (x', y')$, and (x'', y'') be arbitrary elements in A. Then,
$$\begin{aligned}\big[(x, y) + (x', y')\big] + (x'', y'') &= (x + x',\ y + y') + (x'', y'') \\ &= \big((x + x') + x'',\ (y + y') + y''\big) \\ &= \big(x + (x' + x''),\ y + (y' + y'')\big) \\ &= (x, y) + \big[(x', y') + (x'', y'')\big].\end{aligned}$$

Identity Element

The element $(0, 0)$ acts as the identity because for every $(x, y) \in A$,
$$(x, y) + (0, 0) = (x + 0,\ y + 0) = (x, y).$$

Inverse Element

For any $(x, y) \in A$, the element $(-x, -y) \in A$ satisfies

$$(x, y) + (-x, -y) = (x - x,\ y - y) = (0, 0).$$

Commutativity

For any $(x, y), (x', y') \in A$,

$$(x, y) + (x', y') = (x + x',\ y + y') = (x' + x,\ y' + y) = (x', y') + (x, y).$$

Thus, $(A, +)$ is a commutative group.

2. Study of the Multiplication $\times$ on A

The multiplication is defined by

$$(x, y) \times (x', y') = \big(xx',\ x\,y' + x'\,y\big).$$

(a) Commutativity of $\times$

We show that for all $(x, y), (x', y') \in A$:

$$(x, y) \times (x', y') = (x', y') \times (x, y).$$

Compute both sides:

$$\begin{aligned}(x, y) \times (x', y') &= \big(xx',\ x\,y' + x'\,y\big),\\ (x', y') \times (x, y) &= \big(x'x,\ x'\,y + x\,y'\big).\end{aligned}$$

Since $xx' = x'x$ and $x\,y' + x'\,y = x'\,y + x\,y'$, the multiplication is commutative.

(b) Associativity of $\times$

We need to show that for all $(x, y), (x', y'), (x'', y'') \in A$:

$$\big[(x, y) \times (x', y')\big] \times (x'', y'') = (x, y) \times \big[(x', y') \times (x'', y'')\big].$$

Step 1: Compute $\big[(x, y) \times (x', y')\big]$.
Let

$$(x, y) \times (x', y') = \Big(xx',\ x\,y' + x'\,y\Big).$$

Denote this result by (A, B) with:

$$A = xx' \quad \text{and} \quad B = x\,y' + x'\,y.$$

Now,

$$(A, B) \times (x'', y'') = \Big(A\,x'',\; A\,y'' + x''\,B\Big).$$

Substituting back A and B:

$$\begin{aligned}(A, B) \times (x'', y'') &= \Big((xx')x'',\; (xx')y'' + x''(x\,y' + x'\,y)\Big)\\ &= \Big(xx'x'',\; xx'\,y'' + x\,x''\,y' + x'\,x''\,y\Big).\end{aligned}$$

Step 2: Compute $(x, y) \times \big[(x', y') \times (x'', y'')\big]$.
First, compute:

$$(x', y') \times (x'', y'') = \Big(x'x'',\; x'\,y'' + x''\,y'\Big).$$

Denote this as (C, D) where:

$$C = x'x'' \quad \text{and} \quad D = x'\,y'' + x''\,y'.$$

Now,

$$(x, y) \times (C, D) = \Big(xC,\; x\,D + C\,y\Big).$$

Substitute C and D:

$$\begin{aligned}(x, y) \times (C, D) &= \Big(x(x'x''),\; x(x'\,y'' + x''\,y') + (x'x'')y\Big)\\ &= \Big(xx'x'',\; x\,x'\,y'' + x\,x''\,y' + x'\,x''\,y\Big).\end{aligned}$$

Since both computations yield the same result, the multiplication is associative.

(c) Neutral Element for $\times$

We seek an element $e = (a, b) \in A$ such that for every $(x, y) \in A$:

$$(x, y) \times (a, b) = (x, y) \quad \text{and} \quad (a, b) \times (x, y) = (x, y).$$

Compute:

$$(x, y) \times (a, b) = \big(xa,\; x\,b + a\,y\big).$$

For this to equal (x, y) for all (x, y), we require:

$$\begin{cases} xa = x, \\ x\,b + a\,y = y. \end{cases}$$

For the first equation to hold for all x, we must have $a = 1$. Then the second equation becomes:

$$x\,b + y = y \quad \Longrightarrow \quad x\,b = 0 \quad \text{for all } x.$$

Thus, $b = 0$. Hence, the multiplicative identity is:

$$e = (1, 0).$$

(d) Distributivity and the Commutative Ring Structure

To show that $(A, +, \times)$ is a commutative ring, we must verify the distributive laws. For all $(x, y), (x', y'), (x'', y'') \in A$, we need:

$$(x, y) \times \Big[(x', y') + (x'', y'')\Big] = (x, y) \times (x', y') + (x, y) \times (x'', y'')$$

and similarly for the right distributivity.

First, note that:

$$(x', y') + (x'', y'') = (x' + x'',\ y' + y'').$$

Then,

$$\begin{aligned} (x, y) \times (x' + x'',\ y' + y'') &= \Big(x(x' + x''),\ x(y' + y'') + (x' + x'')\,y\Big) \\ &= \Big(xx' + xx'',\ xy' + xy'' + x'y + x''y\Big). \end{aligned}$$

On the other hand, compute individually:

$$\begin{aligned} (x, y) \times (x', y') &= \Big(xx',\ xy' + x'y\Big), \\ (x, y) \times (x'', y'') &= \Big(xx'',\ xy'' + x''y\Big). \end{aligned}$$

Adding these, we get:

$$(xx' + xx'',\ (xy' + x'y) + (xy'' + x''y)) = \Big(xx' + xx'',\ xy' + xy'' + x'y + x''y\Big).$$

Thus, the distributive law holds. Using the commutativity of multiplication, the right distributivity follows similarly.

Since $(A, +)$ is an abelian group, $\times$ is commutative, associative, has the identity $(1, 0)$, and is distributive over $+$, the structure $(A, +, \times)$ is a commutative ring.