Larbi Ben Mhidi University - Oum El Bouaghi-

Faculty of exact sciences, natural and life sciences

Department of Mathematics and Computer Science **Academic year: 2024/2025**

Level: 1st year M+C.S **Exam: Analysis 1** **Duration:1h30**

Exercise 1 (07 pts) (The five questions are independent)

1) Let the set A defined by $A = \left\{1 + \frac{1}{n^2}, n \in \mathbb{N}^*\right\}$. Prove that $\inf A = 1$.

2) Let x be a real number, prove that: $0 \leq E(2x) - 2E(x) \leq 1$.

3) Let the complex number $z_0 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. Calculate z_0^5 and deduce the value of the sum:

$$1 + z_0 + z_0^2 + z_0^3 + z_0^4$$

4) Let f be a function defined on $\mathbb{R}$ by $f(x) = x^3 + 3x + 1$, calculate $(f^{-1})'(1)$.

5) Applying the Mean Value Theorem, prove that: $\forall x \in]0, +\infty[: \frac{x}{1+x} < \ln(x+1) < x$.

Exercise 2 (06 pts)

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence defined by $\begin{cases} u_0 = 0 \\ \forall\, n \in \mathbb{N}: \; u_{n+1} = \frac{u_n^2 + u_n + 1}{u_n + 2}. \end{cases}$

1) Prove that $\forall\, n \in \mathbb{N}: 0 \leq u_n < 1$.

2) Determine the direction of change of the sequence $(u_n)_{n\in\mathbb{N}}$.

3) a) Show that: $\forall\, n \in \mathbb{N}: 1 - u_{n+1} = \left(1 - \frac{1}{u_n+2}\right)(1 - u_n)$.

b) concludes that: $\forall\, n \in \mathbb{N}: 1 - u_{n+1} \leq \frac{2}{3}(1 - u_n)$.

4) Prove that: $\forall\, n \in \mathbb{N}: 1 - u_n \leq \left(\frac{2}{3}\right)^n$, and Deduce $\lim\limits_{n\to\infty} u_n$.

Exercise 3 (07 pts) (The two questions I) and II) are independent)

I) Using L'Hopital's rule, calculate $\lim\limits_{x\to 0} \frac{e^{\sqrt{1+\sin x}} - e}{\tan x}$, Does the function $g: x \to \frac{e^{\sqrt{1+\sin x}} - e}{\tan x}$ accept extension by continuity to 0.

II) Let f be a function defined on $\mathbb{R}$ by $f(x) = \frac{2x}{1+|x|}$.

1) Show that f is bounded on $\mathbb{R}$ (Use the inequality $\forall\, x \in \mathbb{R}: -|x| \leq x \leq |x|$).

2) Write the expression $f(x)$ without the absolute value symbol.

3) Examine the derivability of f at 0, and express $f'(x)$ in terms of x.

4) Show that f is a bijective to $\mathbb{R}$ in the interval $f(\mathbb{R})$, which must be determined.

5) Express $f^{-1}(x)$ in terms of x.

Good luck.

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Department of Mathematics and Computer Science **Academic year: 2024/2025**

Level: 1st year M+C.S **Correction of the exam in analysis 1**

Exercise 1 (07 pts)

1) Let the set A defined by $A = \left\{1 + \frac{1}{n^2}, n \in \mathbb{N}^*\right\}$. Prove that $\inf A = 1$.

we have

$$\inf A = 1 \Leftrightarrow \begin{cases} \forall\, n \in \mathbb{N}^*\ 1 + \dfrac{1}{n^2} \geq 1. \\ \forall \varepsilon; \exists n \in \mathbb{N}^*: 1 + \varepsilon > 1 + \dfrac{1}{n^2} \end{cases}$$ (0.5)

$$\Leftrightarrow \begin{cases} \forall\, n \in \mathbb{N}^*\ \dfrac{1}{n^2} \geq 0. \\ \forall \varepsilon; \exists n \in \mathbb{N}^*: \varepsilon > \dfrac{1}{n^2} \end{cases}$$ (0.5)

$$\Leftrightarrow \begin{cases} \forall\, n \in \mathbb{N}^*\ \dfrac{1}{n^2} \geq 0. \\ \forall \varepsilon; \exists n \in \mathbb{N}^*: 1 < n.\sqrt{\varepsilon} \end{cases}$$ (0.5)

This last proposition is true according to the Archimedean axiom.

2) Let x be a real number, prove that: $0 \leq E(2x) - 2E(x) \leq 1$.

We have

$$2x - 1 < E(2x) \leq 2x \dots\dots\dots\dots\dots (1)$$ (0.25)

In the other hand we have

$$x - 1 < E(x) \leq x \Longrightarrow 2x - 2 < 2E(x) \leq 2x$$ (2x 0.25)

$$\Longrightarrow -2x < -2E(x) \leq -2x + 2 \dots\dots\dots (2)$$ (0.25)

By adding (1) and (2) we get:

$$-1 < E(2x) - 2E(x) < 2,$$ (0.25)

so

$$0 \leq E(2x) - 2E(x) \leq 1.$$ (0.25)

3) Let the complex number $z_0 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. Calculate z_0^5 and deduce the value of the sum $1 + z_0 + z_0^2 + z_0^3 + z_0^4$.

$$z_0^5 = \cos 5\frac{2\pi}{5} + i\sin 5\frac{2\pi}{5}$$ (0.25)

$$= \cos 2\pi + i\sin 2\pi = 1$$ (0.25)

$$1+z_0+z_0^2+z_0^3+z_0^4=\frac{1-z_0^5}{1-z_0}=0$$ (0.5)

4) Let f be a function defined on $\mathbb{R}$ by $f(x)=x^3+3x+1$, calculate $(f^{-1})'(1)$.

We have

$$(f^{-1})'(1)=\frac{1}{f'(x)} \quad \text{where} \quad f(x)=1$$ (0.25)

So

$$f(x)=1 \Leftrightarrow x^3+3x+1=1$$ (0.25)

$$\Leftrightarrow x^3+3x=0$$

$$\Leftrightarrow x=0.$$ (0.25)

And we have $f'(x)=3x^2+3$, so $f'(0)=3$ then (2x0.25)

$$(f^{-1})'(1)=\frac{1}{f'(0)}=\frac{1}{3}$$ (0.25)

5) Applying the Mean Value Theorem, prove that: $\forall x\in\,]0,+\infty[: \frac{x}{1+x}<ln(x+1)<x$.

By applying M.V.T to the function F in the interval

By applying the mean value theorem to the function f in the interval $[0,x]$ we get:

$$f(x)-f(0)=f'(c)(x-0) \quad \text{where} \quad 0<c<x$$ (0.5)

so

$$\ln(x+1)=\frac{1}{1+c}x \quad \text{where} \quad 0<c<x.$$ (0.25)

In the other hand we have

$$0<c<x\Longrightarrow\frac{1}{1+x}<\frac{1}{1+c}<1$$ (0.25)

And for all $x>0$ then:

$$\frac{x}{1+x}<\frac{1}{1+c}x<x$$ (0.25)

So

$$\frac{x}{1+x}< \ln(x+1)<x.$$ (0.25)

Exercise 2 (06 pts)

Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence defined by $\begin{cases} u_0=0 \\ \forall\, n\in\mathbb{N}:\ u_{n+1}=\frac{u_n^2+u_n+1}{u_n+2}. \end{cases}$

1) Prove that $\forall\, n\in\mathbb{N}$: $0\le u_n<1$.

$$0\le u_0<1 \Leftrightarrow 0\le 0<1 \qquad (\text{ is true }).$$ (0.25)

Assume that $\forall\, n \in \mathbb{N}: 0 \le u_n < 1$.

Butting $f(x) = \frac{x^2+x+1}{x+2}$, and $f'(x) = \frac{x^2+4x+1}{(x+2)^2} > 0$ For all $x > 0$. So f is increasing therefore (2x 0.25)

$$0 \le u_n < 1 \implies f(0) \le f(u_n) < f(1)$$ (0.25)

$$\implies \frac{1}{2} \le f(u_n) < 1$$ (0.25)

$$\implies 0 \le u_{n+1} < 1.$$ (0.25)

2) Determine the direction of change of the sequence $(u_n)_{n \in \mathbb{N}}$.

$$u_{n+1} - u_n = \frac{u_n^2 + u_n + 1}{u_n + 2} - u_n$$ (0.25)

$$= \frac{1 - u_n}{u_n + 2} > 0.$$ (0.25)

So (u_n) is increasing

Since $(u_n)_{n \in \mathbb{N}}$ is a bounded increasing sequence, it is therefore convergent. (0.5)

3) a) Show that: $\forall\, n \in \mathbb{N}: 1 - u_{n+1} = \left(1 - \frac{1}{u_n+2}\right)(1 - u_n)$.

we have

$$1 - u_{n+1} = 1 - \frac{u_n^2 + u_n + 1}{u_n + 2}$$ (0.25)

$$= \frac{1 - u_n^2}{u_n + 2}$$ (0.25)

In the other hand we have

$$\left(1 - \frac{1}{u_n + 2}\right)(1 - u_n) = \frac{2 + u_n - 1}{u_n + 2}(1 - u_n)$$ (0.25)

$$= \frac{1 + u_n}{u_n + 2}(1 - u_n)$$ (0.25)

$$= \frac{1 - u_n^2}{u_n + 2}.$$ (0.25)

b) concludes that: $\forall\, n \in \mathbb{N}: 1 - u_{n+1} \le \frac{2}{3}(1 - u_n)$.

we have

$$\forall\, n \in \mathbb{N}: 0 \le u_n < 1 \implies \frac{1}{3} < \frac{1}{u_n + 2} \le \frac{1}{2}$$ (0.25)

$$\implies 1 - \frac{1}{2} < 1 - \frac{1}{u_n + 2} \le 1 - \frac{1}{3}$$ (0.25)

$$\implies \left(1 - \frac{1}{u_n + 2}\right)(1 - u_n) \le \frac{2}{3}(1 - u_n).$$ (0.25)

$$\Longrightarrow 1 - u_{n+1} \leq \frac{2}{3}(1 - u_n).$$ (0.25)

4) Prove that: $\forall\, n \in \mathbb{N}: 1 - u_n \leq \left(\frac{2}{3}\right)^n$.

By induction

$$1 - u_0 \leq \left(\frac{2}{3}\right)^0 \Leftrightarrow 1 \leq 1 \qquad (\text{ is true }).$$ (0.25)

Assume that $1 - u_n \leq \left(\frac{2}{3}\right)^n$. (0.25)

Since $1 - u_{n+1} \leq \frac{2}{3}(1 - u_n)$ then $1 - u_{n+1} \leq \frac{2}{3}\left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^{n+1}$. (0.25)

Deduce $\lim\limits_{n\to\infty} u_n$.

Since $\forall\, n \in \mathbb{N}: 0 \leq 1 - u_n \leq \left(\frac{2}{3}\right)^n$ and $\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$, then $\lim\limits_{n\to\infty} u_n =$ ~~0.~~ =1. (2x0.25)

<u>Exercise 3 (07 pts)</u>

I) Using L'Hopital's rule, calculate $\lim\limits_{x\to 0}\frac{e^{\sqrt{1+\sin x}} - e}{\tan x}$.

$$\lim_{x\to 0}\frac{e^{\sqrt{1+\sin x}} - e}{\tan x} = \lim_{x\to 0}\frac{\left(e^{\sqrt{1+\sin x}} - e\right)'}{(\tan x)'} = \lim_{x\to 0}\frac{e^{\sqrt{1+\sin x}}\frac{\cos x}{2\sqrt{1+\sin x}}}{\frac{1}{\cos^2 x}} = \frac{e}{2}$$ (0.25+ 0.5+ 0.25)

Does the function $g: x \to \frac{e^{\sqrt{1+\sin x}} - e}{\tan x}$ accept extension by continuity at 0.

Since $\lim\limits_{x\to 0} g(x) = \frac{e}{2} < \infty$ then g accept an extension by continuity at 0 and we have (0.25)

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } \; x \neq 0 \\ \frac{e}{2} & \text{if } \; x \end{cases}$$ ~~≠ 0.~~ =0. (0.25)

II) Let f be a function defined on $\mathbb{R}$ by $f(x) = \frac{2x}{1+|x|}$.

1) Show that f is bounded on $\mathbb{R}$ (Use the inequality $\forall\, x \in \mathbb{R}: -|x| \leq x \leq |x|$).

Since $\forall\, x \in \mathbb{R}: -|x| \leq x \leq |x| \Longrightarrow -2|x| \leq 2x \leq 2|x|$ (0.25)

$$\Longrightarrow \frac{-2|x|}{1+|x|} \leq \frac{2x}{1+|x|} \leq \frac{2|x|}{1+|x|}$$ (0.25)

$$\Longrightarrow -2 + \frac{2}{1+|x|} \leq \frac{2x}{1+|x|} \leq 2 - \frac{2}{1+|x|}$$ (0.25)

$$\Longrightarrow -2 \leq f(x) \leq 2.$$ (0.25)

2) Write the expression $f(x)$ without the absolute value symbol.

$$f(x)=\frac{2x}{1+|x|}=\begin{cases}\dfrac{2x}{1+x} & \text{if } x\geq 0 \\ \dfrac{2x}{1-x} & \text{if } x<0\end{cases}$$ (0.25) (0.25)

3) Examine the derivability of f at 0, and express $f'(x)$ in terms of x.

$$\lim_{\substack{< \\ x\to 0}}\frac{f(x)-f(0)}{x-0}=\lim_{\substack{< \\ x\to 0}}\frac{\frac{2x}{1-x}-0}{x-0}=\lim_{\substack{< \\ x\to 0}}\frac{2}{1-x}=2=f'(0-0)$$ (0.5)

$$\lim_{\substack{> \\ x\to 0}}\frac{f(x)-f(0)}{x-0}=\lim_{\substack{> \\ x\to 0}}\frac{\frac{2x}{1+x}-0}{x-0}=\lim_{\substack{> \\ x\to 0}}\frac{2}{1+x}=2=f'(0+0).$$ (0.5)

So f is derivable at 0 and $f'(0)=2$.

$$f(x)=\frac{2x}{1+|x|}=\begin{cases}\dfrac{2}{(1+x)^2} & \text{if } x\geq 0 \\ \dfrac{2}{(1-x)^2} & \text{if } x<0\end{cases}$$ (0.5) (0.5)

4) Show that f is a bijective from $\mathbb{R}$ towards the interval $f(\mathbb{R})$, which must be determined.

We have $\forall\, x\in\mathbb{R}$: $f'(x)\geq 0$ then f is continuouse and strictly increasing on $\mathbb{R}$, so f is a bijective to $\mathbb{R}$ in $f(\mathbb{R})$. (2x 0.25)

Since $\lim\limits_{x\longrightarrow+\infty} f(x)=2$ and $\lim\limits_{x\longrightarrow+\infty} f(x)=-2$, then $f(\mathbb{R})=]-2,2[$. (2x 0.25)

5) Express $f^{-1}(x)$ in terms of x.

For $x\geq 0$ and $0\leq y<2$.

$$f(x)=y \iff \frac{2x}{1+x}=y$$ (0.25)

$$\iff x=\frac{y}{2-y}$$ (0.25)

For $x<0$ and $-2<y<0$.

$$f(x)=y \iff \frac{2x}{1-x}=y$$ (0.25)

$$\iff x=\frac{y}{2+y}.$$ (0.25)

So

$$f^{-1}(x)=\begin{cases}\dfrac{x}{2-x} & \text{if } 0\leq x<2 \\ \dfrac{x}{2+x} & \text{if } -2<x<0\end{cases}$$