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Teacher: $Y.SOULA$ Module: *Algebra* 1

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Correction typical

Exercise 1. (7 *points*)

1)

a) the negation is given by

$\exists x \in \mathbb{R} : 2x \leq x$..(1$pt$)

b) the negation of proposition (b) is given by

$\exists x \in \mathbb{R} : x > 0 \wedge 2x \leq x$..(1$pt$)

c) the negation is given by

$\forall n \in \mathbb{N}, \quad 5n + 11 \neq 3n + 14$..(1$pt$)

d) the negation is given by

$\exists \varepsilon \in \mathbb{R} \quad \varepsilon > 0 \wedge \forall n \in \mathbb{N}^* : \frac{1}{n} \geq \varepsilon$...(1$pt$)

2)

proposition (**a**) is false because its negation is true.

proposition (**b**) is true. For the proof, we proceed as follows

let $x \in \mathbb{R}$,

suppose that $x > 0$ hence $2x = x + x > x$

this completes the proof..(3pt)

3)

proposition (**c**) is false. Solving the equation $5n + 11 = 3n + 14$

we find that the only solution is $n = \frac{3}{2}$, or $\frac{3}{2} \notin \mathbb{N}$

Therefore

$\forall n \in \mathbb{N} : \quad 5n + 11 \neq 3n + 14$

is true, or proposition " $\exists n \in \mathbb{N}, \quad 5n + 11 = 3n + 14$" is false.

Exercise 2.

1) $\Re$ is reflexive because:

$\forall x \in \mathbb{R},\ e^x \leq e^x$, So, $x \Re x$...(1.5pt)

2) $\Re$ is not symmetric because:

$\exists\ 0, 1 \in \mathbb{R},\ e^0 = 1 \leq e^1$ but $e^1 \nleq e^0$..(1.5pt)

3) $\Re$ is anti-symmetric because:

$$\begin{aligned}\forall x, y \in \mathbb{R},\ x \Re y \wedge y \Re x &\Leftrightarrow e^x \leq e^y \wedge e^y \leq e^x \\ &\Rightarrow e^x = e^y \Leftrightarrow x = y\end{aligned}$$ ....(1.5pt)

4) $\Re$ is transitive because:

$\forall x, y, z \in \mathbb{R}, x \Re y \wedge y \Re z \Leftrightarrow e^x \leq e^y \wedge e^y \leq e^z \Rightarrow e^x \leq e^z \Leftrightarrow x \Re z$.............(1.5$pt$)

In conclusion $\Re$ is an order relation but it is not an equivalence relation.

Exercise 3.

1) $(x, y) + (x^{'}, y^{'}) = (x + x^{'}, y + y^{'}) \in A$ so the law $+$ is internal..................(0.5pt)

Let (x, y), $(x^{'}, y^{'})$, $(x^{''}, y^{''}) \in A$, we have:

$$\begin{aligned}(x, y) + [(x^{'}, y^{'}) + (x^{''}, y^{''})] &= (x, y) + (x^{'} + x^{''}, y^{'} + y^{''}) \\ &= (x + (x^{'} + x^{''}), y + (y^{'} + y^{''}))\end{aligned}$$

$$= ((x+x^{'})+x^{''}, (y+y^{'})+y^{''})$$
$$= [(x,y)+(x^{'},y^{'})]+(x^{''},y^{''}).$$

So the law $+$ is associative..(1pt)

Let (x,y), $(x^{'},y^{'}) \in A$, we have:

$$\begin{aligned}(x,y)+(x^{'},y^{'}) &= (x+x^{'}, y+y^{'})\\ &= (x^{'}+x, y^{'}+y)\\ &= (x^{'},y^{'})+(x,y).\end{aligned}$$

So the $+$ law is commutative..(1pt)

Let $(a,b) \in A = \mathbb{R}\times\mathbb{R}$, such that $(x,y)+(a,b)=(x,y)$,
it is clear that $(a,b)=(0,0)$ is the unique neutral element..............................(1pt)

Let $(x^{'},y^{'}) \in A = \mathbb{R}\times\mathbb{R}$ such that $(x,y)+(x^{'},y^{'})=(0,0)$
this is equivalent to:

$$(x+x^{'}, y+y^{'}) = (0,0) \Leftrightarrow \begin{cases} x+x^{'}=0 \\ y+y^{'}=0 \end{cases} \Leftrightarrow \begin{cases} x^{'}=-x \\ y^{'}=-y \end{cases}$$

So the symmetric of (x,y) is $(-x,-y)$..(1pt)

So $(A,+)$ is a commutative group.

2)

a) $\forall\ (x,y),(x^{'},y^{'}) \in A : (x,y)\star(x^{'},y^{'}) = (xx^{'}, xy^{'}+x^{'}y)$

$$\begin{aligned} &= (x^{'}x, y^{'}x+yx^{'})\\ &= (x^{'}x, x^{'}y+xy^{'})\\ &= (x^{'},y^{'})\star(x,y)\end{aligned}$$

So $\star$ is commutative..(0.5pt)

b) $\forall\ (x,y),(x^{'},y^{'}),\ (x^{''},y^{''}) \in A :$

$$\begin{aligned}[(x,y)\star(x^{'},y^{'})]\star(x^{''},y^{''}) &= (xx^{'}, xy^{'}+x^{'}y)\star(x^{''},y^{''})\\ &= (xx^{'}x^{''}, xx^{'}y^{''}+x^{''}(xy^{'}+x^{'}y))\\ &= (xx^{'}x^{''}, xx^{'}y^{''}+x^{''}xy^{'}+x^{''}x^{'}y))\text{.........(1)}\end{aligned}$$

$$\begin{aligned}(x,y)\star[(x^{'},y^{'})\star(x^{''},y^{''})] &= (x,y)\star(x^{'}x^{''}, x^{'}y^{''}+x^{''}y^{'})\\ &= (xx^{'}x^{''}, x(x^{'}y^{''}+x^{''}y^{'})+x^{'}x^{''}y)\\ &= (xx^{'}x^{''}, xx^{'}y^{''}+xx^{''}y^{'}+x^{'}x^{''}y)\\ &= (xx^{'}x^{''}, xx^{'}y^{''}+x^{''}xy^{'}+x^{''}x^{'}y)\text{.........(2)}\end{aligned}$$

From (1) and (2) we find that:

$[(x,y)\star(x^{'},y^{'})]\star(x^{''},y^{''}) = (x,y)\star[(x^{'},y^{'})\star(x^{''},y^{''})]$

So the law $\star$ is associative...(0.5pt)

c)

Let $(e,f)\in A$ such that for all $(x,y)\in A$,

$$(x,y)\star(e,f\ ) = (x,y) \Leftrightarrow \begin{cases} xe = x \\ xf+ey = y \end{cases} \Leftrightarrow \begin{cases} e=1 \\ f=0 \end{cases}$$

$(1,0)\in A$ is the neutral element of A for the law $\star$...(0.5pt)

d)
All the properties for a set with two laws to be a ring are in the previous questions except the distributivity of $\star$ with respect to addition $(+)$ (to the left or to the right since the law $\star$ is commutative, it is moreover this commutativity which makes the ring commutative).

$$\begin{aligned}(x,y)\star[(x^{'},y^{'})+(x^{''},y^{''})] &= (x,y)\star(x^{'}+x^{''},y^{'}+y^{''})\\ &= (x(x^{'}+x^{''}),\, x(y^{'}+y^{''})+(x^{'}+x^{''})y)\\ &= (xx^{'}+xx^{''},\, xy^{'}+xy^{''}+x^{'}y+x^{''}y)\\ &= (xx^{'}+xx^{''},\, xy^{'}+x^{'}y+xy^{''}+x^{''}y)\\ &= (xx^{'},\, xy^{'}+x^{'}y)+(xx^{''},xy^{''}+x^{''}y)\\ &= [(x,y)\star(x^{'},y^{'})]+[(x,y)\star(x^{''},y^{''})]..................(1pt)\end{aligned}$$

And here $(A,+,\star)$ is a commutative ring.