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Faculty of SENV, Department of MI
Final Exam of First Semestre 2024/ 2025
Time 1H.30m

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Exercise 1 (10 marks)
Let A be the followin matrix:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

1. Prove that $\ker(A+I) = vect((-4, 10, 9))$.
2. Without calculation of the characteristic polynomial, find the eigenvalues of A.
3. Deduce the characteristic and the minimal polynomials of A.
4. Deduce the inverse of A.
5. Deduce the power of A.

Exercise 2 (4 marks)
Let $0 < \theta < \pi$. Find all the invariant vector lines of the matrix

$$B = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Exercise 3 (6 marks)
Let the following:

$$C = \begin{pmatrix} -1 & 2 & 1 \\ 2 & -1 & -1 \\ -4 & 4 & 3 \end{pmatrix},$$

and

$$\ker(C+I) = vect\{(1,-1,2)\}, \ker(C-I) = vect\{(1,0,2),(0,-1,2)\}$$

1. Calculate C^n.
2. Let $U_0 = (-2, 4, 1)$, deduce $U_{n+1} = CU_n$ in fonction of n.

Solution of the exam

Exercise 1

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

1. To prove that $\ker(A+I) = vect\{(-4,10,9)\}$ it is equivalent to prove that $\dim\ker(A+I) = 1$ and $(-4,10,9) \in \ker(A+I)$

   $$\ker(A+I) = \left\{(x,y,z) \in \mathbb{R}^3 |\, (A+I)(x,y,z) = 0\right\}.$$

   That means to solve the following system:

   $$\left\{ \begin{array}{rcl} 2x - y + 2z & = & 0 \\ 5x + 2y & = & 0 \end{array} \right.$$

   Which gives

   $$\ker(A+I) = \left\{ x\left(1, -\frac{5}{2}, -\frac{9}{4}\right),\ x \in \mathbb{R} \right\}$$

   Therefore $\dim\ker(A+I) = 1$. Since $(-4,10,9) = -4 \times \left(1, -\frac{5}{2}, -\frac{9}{4}\right)$, then $(-4,10,9) \in \ker(A+I)$. Or

   $$\begin{pmatrix} 2 & -1 & 2 \\ 5 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -4 \\ 10 \\ 9 \end{pmatrix} = \begin{pmatrix} -8-10+18 \\ -20+20 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

2. Since $\ker(A+I) \neq 0$, then $\lambda_1 = -1$ is an eigenvalue of A. As A is of order 3, then we have three eigenvalues $\lambda_1 = -1$, λ_2, λ_3 of A in the complex field satisfied the following system:

   $$\left\{ \begin{array}{rcl} \lambda_1 + \lambda_2 + \lambda_3 & = & tr(A) \\ \lambda_1\lambda_2\lambda_3 & = & \det A \end{array} \right.$$

   which gives

   $$\left\{ \begin{array}{rcl} \lambda_2 + \lambda_3 & = & 2 \\ \lambda_2\lambda_3 & = & 6 \end{array} \right.$$

Then to find $\lambda_2$and λ_3, it is sufficient to solve the equation

$$X^2 - 2X + 6 = 0.$$

That gives

$$\lambda_2 = 1 + i\sqrt{5} \text{ and } \lambda_3 = 1 - i\sqrt{5}.$$

3. The characteristic polynomial

   Since the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1 + i\sqrt{5}$ and $\lambda_3 = 1 - i\sqrt{5}$ and the order of the matrix is 3, then the characteristic polynomial is:

$$\begin{aligned} C_A(X) &= (X - \lambda_1)(X - \lambda_2)(X - \lambda_3) \\ &= (X + 1)\left(X - 1 - i\sqrt{5}\right)\left(X - 1 + i\sqrt{5}\right). \end{aligned}$$

   Since all the eigenvalues are of multiplicity 1, then the characteristic and the minimal polynomials are identical, i.e.

$$C_A(X) = m_A(X).$$

4. The inverse of A.

   Since

$$m_A(X) = (X + 1)\left(X - 1 - i\sqrt{5}\right)\left(X - 1 + i\sqrt{5}\right) = X^3 - X^2 + 4X + 6,$$

   then

$$m_A(A) = 0,$$

   which yields to

$$\frac{-1}{6}\left(A^2 - A + 4I\right)A = I \Rightarrow A^{-1} = \frac{-1}{6}\left(A^2 - A + 4I\right).$$

   Thus

$$A^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{5}{6} & \frac{1}{6} & -\frac{5}{3} \\ 0 & 0 & -1 \end{pmatrix}$$

5. The power of A

$$A^n = P \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & \left(1 + i\sqrt{5}\right)^n & 0 \\ 0 & 0 & \left(1 - i\sqrt{5}\right)^n \end{pmatrix} P^{-1}$$

Exercise 2

The invariant vector lines of the matrix are the subspaces of $\mathbb{R}^3$ or $\mathbb{C}^3$ of dimension 1. That means for any (x, y, z) in that subspace, there exists a scalar λ such that $(B - \lambda I)(x, y, z) = 0$. That means the invariant subspaces are $\ker(B - \lambda I)$ of dimension 1. So let us find λ. That means to calculate $\det(XI - B) = C_B(X)$, which gives:

$$C_B(X) = (X + 1)\left(X^2 - (2\cos\theta)X + 1\right)$$

Since $\lambda = -1$ is an eigenvalue of multiplicity 1, then one of the vector lines are $\ker(B + I)$. Since $0 < \theta < \pi$, then $4\left((\cos\theta)^2 - 1\right) < 0$. That means $X^2 - (2\cos\theta)X + 1$ has no roots on $\mathbb{R}$. Therefore, we have only one vector line

$$\ker(B + I) = vect\{(0, 0, 1)\},$$

while on $\mathbb{C}$, in addition to that, we have

$$\ker(B - (\cos\theta + i\sin\theta)I) = vect\{(-i, 1, 0)\} \text{ and } \ker(B - (\cos\theta - i\sin\theta)I) = vect\{(i, 1, 0)\}.$$

Exercise 3

To answer to the questions, first of all, we have to prove that C is diagonable and to put it in the diagonal form.

Since we have

$$\ker(C + I) = vect\{(1, -1, 2)\}, \ker(C - I) = vect\{(1, 0, 2), (0, -1, 2)\},$$

then the eigenvalues of C are

$$\lambda_1 = -1, \lambda_2 = \lambda_3 = 1$$

and the eigenvectors are

$$(1, -1, 2), (1, 0, 2), (0, -1, 2)$$

Since the eigenvectors $(1, 0, 2), (0, -1, 2)$ are lineary independent and they are independent of $(1, -1, 2)$ (because the associated eigenvalues are different), then the set

$$\{(1, -1, 2)\} \cup \{(1, 0, 2), (0, -1, 2)\} = \{(1, -1, 2), (1, 0, 2), (0, -1, 2)\}$$

is a basis of $\mathbb{R}^3$. Therefore the matrix C is diagonable: $C = PDP^1$, where

$$D = diag\,(-1,1,1) \text{ and } P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 2 & 2 & 2 \end{pmatrix}$$

Thus

$$C^n = PD^nP^1 = P\begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1}$$

1. Consequently, we get

$$C^n = \begin{cases} I \text{ for } n \text{ even} \\ C \text{ for } n \text{ odd} \end{cases}$$

Indeed, for $n = 2k+1$, $C^n = C^{2k+1} = C^{2k}C$. Since $C^n = I$ for n even, then $C^{2k} = I$. Therefore, $C^{2k+1} = C$.

2. Let

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = C \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

Then

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = C^n \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} = \begin{cases} \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} & \text{for } n \text{ even} \\ \begin{pmatrix} 11 \\ -9 \\ 27 \end{pmatrix} & \text{for } n \text{ odd} \end{cases}$$