L'arbi Ben M'hidi University

Faculty: Exact Sciences, Natural and Life sciences
Department : Mathematics and Computer Science.
Level: Second year of Bachelor Mathematics

In: 14/01/2025
Duration: 1h30min

Mathematical logic exam

Exercise 1: (6 pts)
Are the following statements true or false? If true, give an example, otherwise correct it.
1* Two sets are equipotent if there is a bijective application between them.
2* A set has the the cardinality of the continuum if it is equipotent with $\mathbb{N}$.
3* A set is said to be denumerable if and only if it is equipotent with $\mathbb{R}$.
4* A set is said to be infinite if and only if it is equipotent with $\mathbb{N}_n$ such that

$$\mathbb{N}_n = \{0, 1, ., n-1\}$$

Exercise 2: (4 pts)

The Fibonacci sequence is given by $\begin{cases} F_0 = 0 \\ F_1 = 1 \\ \forall n \in \mathbb{N} : F_{n+2} = F_{n+1} + F_n \end{cases}$

Let $\varphi = \frac{1+\sqrt{5}}{2}$ and $\varphi' = \frac{1-\sqrt{5}}{2}$ (φ is called the golden ratio). We have φ and φ' are solutions of the equation $x^2 - x - 1 = 0$
Question: By absurdity and well ordering proof show that for all $n \geq 1$ we have $F_n \leq \varphi^{n-1}$.

Exercise 3: (4 pts)
1/Use the truth tables method to determine whether the following formula

$$(\rceil P \vee Q) \wedge (Q \Longrightarrow (\rceil R \wedge \rceil P)) \wedge (P \vee R)$$

is an antilogy.
2/ Let α be an antilogy and β any formula. Show that $\alpha \Longrightarrow \beta$ is a tautology.

Exercise 4: (3 pts)

Show that : $\forall n \in \mathbb{N}$; 7 divides $3^{2n+1} + 2^{n+2}$

Exercise 5: (3 pts)
Give a logical paradox and explain why it is a paradox (by giving the text, the question posed and the possible answers).

<u>Solution</u>

<u>EX01</u>:

1⟶ True(0.75) , Example (0.75)

2⟶ False(0.75) Correction : A set has the cardinality of the continum if it is equipotent with $\mathbb{R}$. (0.75)

3⟶ False(0.75) Correction: A set is said to be denumerable if and only if it is equipotent with $\mathbb{N}$. (0.75)

4⟶ False(0.75) Correction: A set is said to be <u>finite</u> if and only if it is equipotent with $\mathbb{N}_n$ (0.75)

<u>EX02:</u>

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ \forall n \in \mathbb{N} : F_{n+2} = F_{n+1} + F_n \ldots\ldots (*) \end{cases}$$

By absurdity and well ordering proof showing that $\forall n \geq 1 : F_n \leq \varphi^{n-1}$.

Assume that it exits a set $C \neq \emptyset$, such that $C = \{n \in \mathbb{N}, F_n > \varphi^{n-1}\}$ (0.5)

We have $C \subset \mathbb{N}$, by well ordering principle $\exists n_0 \in C, n_0$ is the minimum of C.(it means that $\forall n \in C : n_0 \leq n$) (0.5)

Then $F_{n_0} > \varphi^{n_0-1}$, in other part we have $F_1 \leq \varphi^0 = 1$, so $1 \notin C$ and $n_0 - 1 \notin C$ as a result $F_{n_0-1} \leq \varphi^{n_0-2}$.... (0.5)

And $F_0 = 0 \leq \varphi^{-1}$ is also true, so $F_{n_0-2} \leq \varphi^{n_0-3}$.... (0.5)

By $(*)$ we get $F_{n_0} = F_{n_0-1} + F_{n_0-2}$.... (0.25)

$\Longrightarrow F_{n_0} \leq \varphi^{n_0-2} + \varphi^{n_0-3}$... (0.25)

$\Longrightarrow F_{n_0} \leq \varphi^{n_0-3}(\varphi + 1)$... (0.25) $\quad [**]$

But φ is a solution for the equation $x^2 - x - 1 = 0$ so $\varphi^2 - \varphi - 1 = 0 \Longrightarrow \varphi^2 = \varphi + 1$.. (0.5) . $[***]$

Remplace $[***]$ in $[**]$ we get $F_{n_0} \leq \varphi^{n_0-3}\varphi^2$ then $F_{n_0} \leq \varphi^{n_0-1}$... (0.5)

we have a contradiction then $C = \emptyset$ and $n \geq 1$ we have $F_n \leq \varphi^{n-1}$...(0.25)

<u>EX03:</u>

	P	Q	R	$\rceil P$	$\rceil R$	$\rceil P \vee Q$	$\rceil R \wedge \rceil P$	$Q \Longrightarrow (\rceil R \wedge \rceil P)$	$P \vee R$	$(\rceil P \vee Q) \wedge (Q \Longrightarrow (\rceil R \wedge \rceil P)) \wedge ($
	1	1	1	0	0	1	0	0	1	0
	1	1	0	0	1	1	0	0	1	0
	1	0	1	0	0	0	0	1	1	0
1/	1	0	0	0	1	0	0	1	1	0
	0	1	1	1	0	1	0	0	1	0
	0	1	0	1	1	1	1	1	0	0
	0	0	1	1	0	1	0	1	1	1
	0	0	0	1	1	1	1	1	0	0

This formula is not antilogy (0.25)

	α	β	$\alpha \Longrightarrow \beta$	
2/	0	1	1	So in two cases $\alpha \Longrightarrow \beta$ is true, then $\alpha \Longrightarrow \beta$ is a tautology.....(0.75)
	0	0	1	

<u>EX04:</u>

Show that : $\forall n \in \mathbb{N}$; 7 divides $3^{2n+1} + 2^{n+2}$.....$p(n)$ by recurrence proof

*For $n = 0$; 7 divides $3^{0+1} + 2^{0+2} = 7$ is true...(0.5)

*Assume that $p(n)$ is true and showing that $p(n+1)$ is also true

$p(n)$ is true means that $\exists k \in \mathbb{N} : 3^{2n+1} + 2^{n+2} = 7k$... (0.5) ... $[*]$

For $n+1: 3^{2n+3}+2^{n+3} = 3^{2n+1}\times 3^2 + 2^{n+2}\times 2 ... (0.25)$

$$= 3^{2n+1}\times 9 + 2^{n+2}\times 2 ... (0.25)$$
$$= 3^{2n+1}\times (7+2) + 2^{n+2}\times 2 ... (0.25)$$
$$= 3^{2n+1}\times 7 + \left(2^{n+2}+3^{2n+1}\right)\times 2 ... (0.25)$$
$$= 3^{2n+1}\times 7 + (7k)\times 2 ... (0.25) \text{ by } [*]$$
$$= 7\left(3^{2n+1}+2k\right) ... (0.25)$$

but $3^{2n+1}+2k \in \mathbb{N}$ so we can put $3^{2n+1}+2k = k'$

Then $3^{2n+3}+2^{n+3} = 7k'$.. (0.25) which means 7 divides $3^{2n+3}+2^{n+3}$.and $p(n+1)$ is true and by recurrence $p(n)$ is true....(0.25)

EX05:

text (1)

question (1)

answer 1(0.5)

answer 2 (0.5)