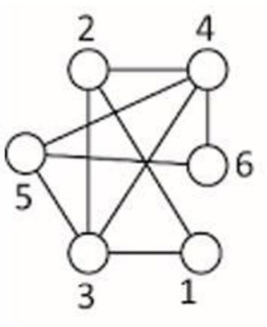
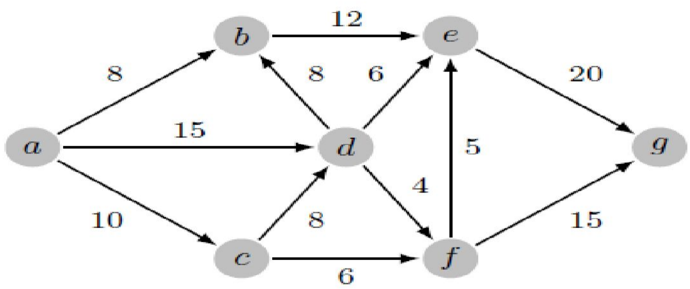
Graph Theory Exam

Exercise n°=1 : (4 pts)

Let be the following graph G :  	Determine **(justifying your answer)** whether the graph G is a planar graph. **(1 pt)** If yes, a. give its planar representation, **(1 pt)** b. determine the number of faces, **(1 pt)** c. and give the corresponding dual graph G^*. **(1 pt)**

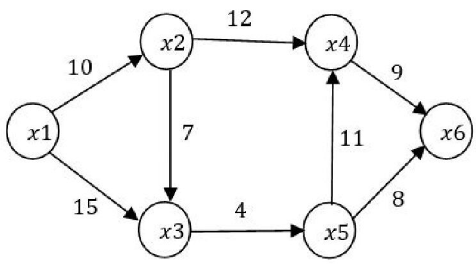
Exercise n°=2 : (7 pts)

Consider the following weighted network $G = (X, U, C)$:



- Using the Ford-Fulkerson algorithm, give **a maximum flow** for this transport network and calculate **a minimum cut**.

Exercise n°=3 : (5 pts)

Let be the following graph G :  	Determine a minimal weight path from vertex $\boldsymbol{x1}$ to each of the other vertices of the graph G, indicating the different steps.

Turn the sheet

Last name :………………………	First name :……………………………	Group :………	Mark (MCQ) :	4

comprehension questions (MCQ) : (4 pts)

Check the correct answer(s) in the following :

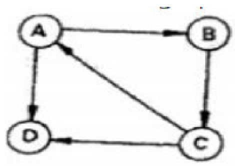
1) **Let *X* be the adjacency matrix of a graph G with no self loops. The entries along the principal diagonal of X are :**

	all ones
	all zeros
	both zeros and ones
	different

2) **The in-degree of a vertex in a directed graph is:**

	The number of vertices connected to it
	The number of edges leaving it
	The number of edges coming into it
	The sum of its neighbors degrees

3) **Consider the graph shown in the figure below :**



Which of the following is a valid strongly component ?

	$\{A, C, D\}$
	$\{A, C, B\}$
	$\{A, D, B\}$
	$\{B, C, D\}$

4) **A graph is considered complete if :**

	All its edges are collinear
	All its vertices are adjacent to each other
	It is composed of straight lines
	It is oriented

5) **Chain length is :**

	The number of edges that compose it
	The number of vertices that compose it
	The number of graphs that compose it
	The number of matrices that compose it

6) **An elementary path can pass through the same arc several times :**

	True
	False

7) **In a graph G there is one and only one path between every pair of vertices then G is a :**

	Path
	Walk
	Circuit
	Tree

8) **A graph in which all nodes are of equal degree, is known as :**

	Multigraph
	Non regular graph
	Regular graph
	Complete graph

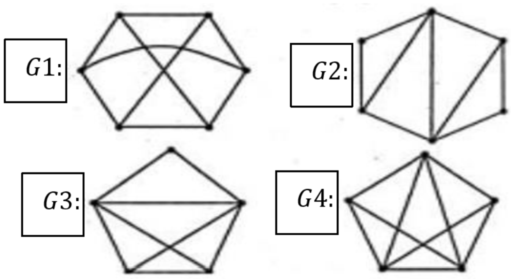
9) **A vertex-edge matrix is composed of elements whose values can be :**

	0
	1
	-1
	'True' ou 'False'

10) **The maximum number of edges in a bipartite graph on 12 vertices is :**

	12
	24
	36
	48

11) **Which one of the following graphs is NOT planar ?**



	$G1$
	$G2$
	$G3$
	$G4$

12) **Let G be a simple connected planar graph with 13 vertices and 19 edges. Then, the number of faces in the planar embedding of the graph is :**

	6
	8
	9
	13

13) **If the graph is with triangle, then we apply Euler's property 1 :**

	$m \leq 6 \times n - 3$
	$m \leq 3 \times n - 6$
	$m \leq 2 \times n - 4$
	$m \leq 4 \times n - 2$

14) **A graph is a tree if and only if :**

	Is planar
	Contains a circuit
	Is minimally
	Is completely connected

15) **The empty subgraph of a graph contains :**

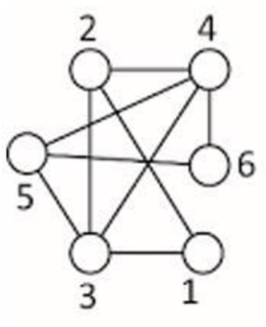
	No vertices and no edges
	No vertices but some edges
	Some vertices but no edges
	All the vertices but no edges

Larbi Ben Mhidi University - Oum El Bouaghi
Faculty of Exact Sciences, Natural and Life Sciences
Department of Mathematics and Computer Science

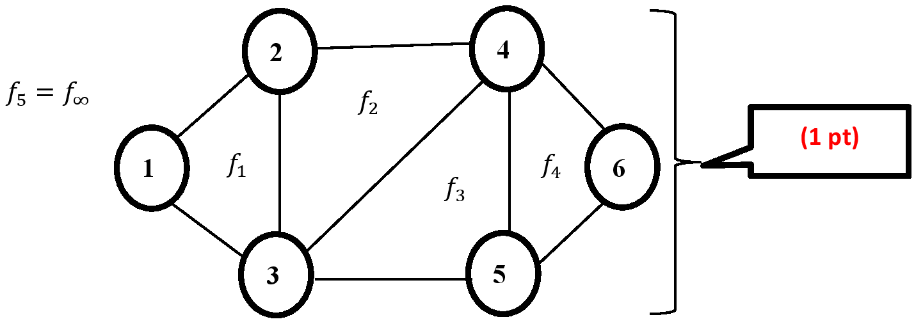
2nd Year Bachelor of Computer Science – 3rd Semester
Saturday 18/01/2025
Duration : 1 H 30 mn

Graph Theory Exam

Exercise n°=1 : (4 pts)

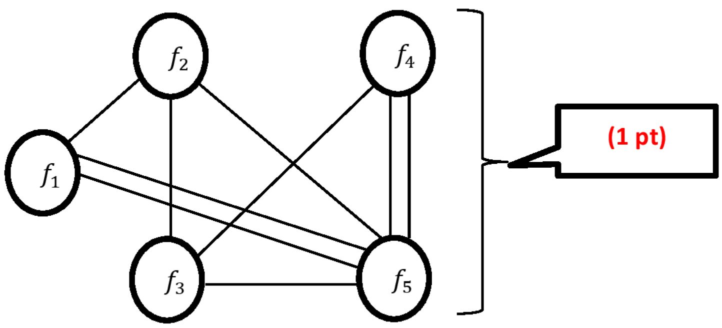
Let be the following graph G :  	Determine **(justifying your answer)** whether the graph G is a planar graph. **(1 pt)** If yes, a. give its planar representation, **(1 pt)** b. determine the number of faces, **(1 pt)** c. and give the corresponding dual graph G^*. **(1 pt)**

Yes, because it can be represented without intersection as shown in the following figure :



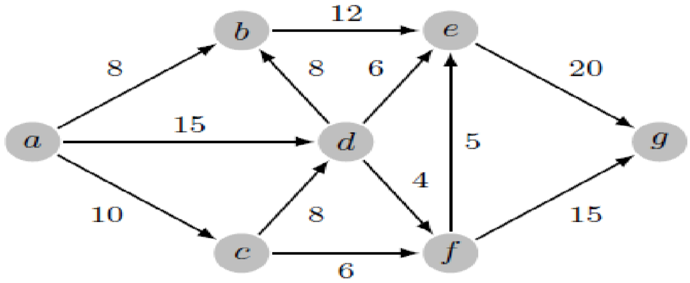
The graph has a triangle, so we apply Euler's property 1. We have : $\boldsymbol{m \leq 3 \times n - 6}$ $\frac{2 \times 2 + 3 \times 2 + 4 \times 2}{2} \leq 3 \times 6 - 6 => 9 \leq 12 => TRUE$ so the graph is planar.	**(1 pt)**
Let's apply Euler's formula : $n - m + f = 2 \rightarrow f = 2 - n + m$ $f = 2 - 6 + 9 = 5\ faces$	**(1 pt)**

The corresponding dual graph G^* :



Exercise n°=2 : **(7 pts)**

Consider the following weighted network $G = (X, U, C)$:

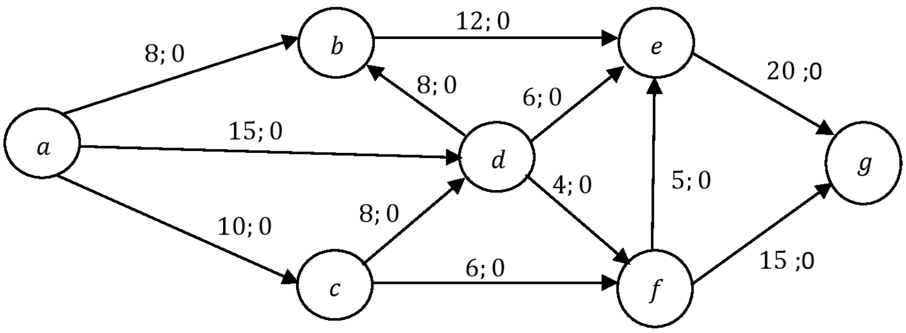


- Using the Ford-Fulkerson algorithm, give **a maximum flow** for this transport network and calculate **a minimum cut**.

Application of the Ford-Fulkerson algorithm for finding the maximum flow (3.5 pts)

Initialization :

$k = 0$; $\varphi^k = 0$; $A = \{\boldsymbol{a}\}$

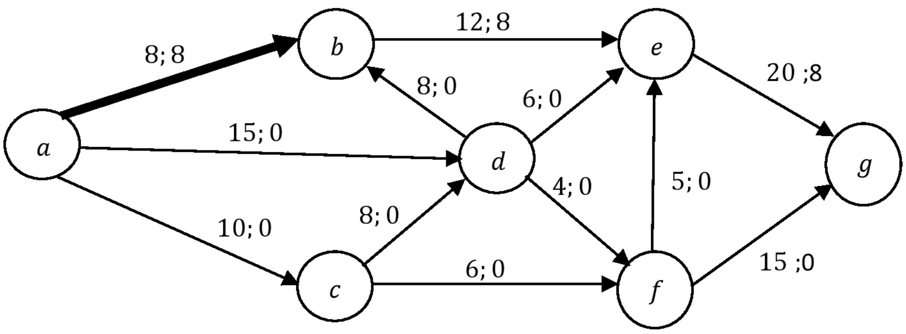


Iteration 1 :

$A = \{a, b, e, g\}$

$\varepsilon = min\{8-0, 12-0, 20-0\} = min\{8, 12, 20\} = 8$

$\varphi^1 = \varphi^0 + \varepsilon = 0 + 8 = 8$

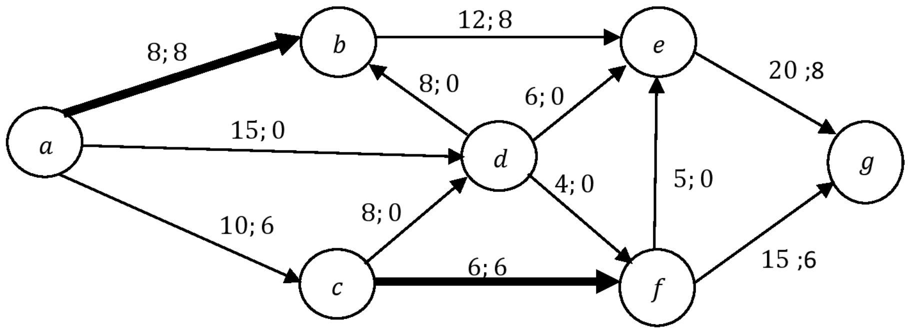


Iteration 2 :

$A = \{a, c, f, g\}$

$\varepsilon = min\{10-0, 6-0, 15-0\} = min\{10, 6, 15\} = 6$

$\varphi^2 = \varphi^1 + \varepsilon = 8 + 6 = 14$

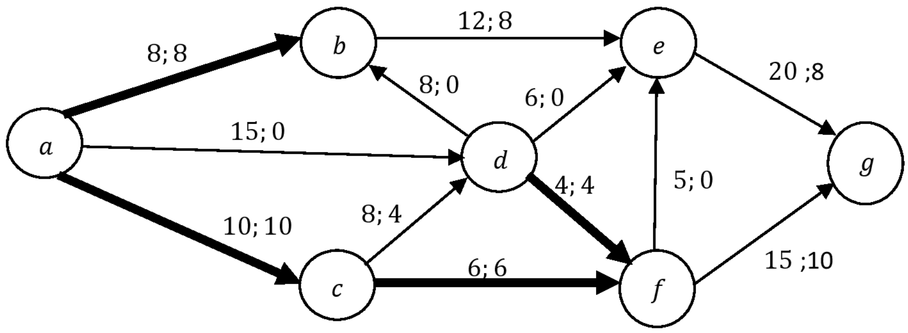


Iteration 3 :

$\mathrm{A} = \{a, c, d, f, g\}$

$\varepsilon = min\{10 - 6{,}8 - 0{,}4 - 0{,}15 - 6\} = min\{4{,}8{,}4{,}9\} = 4$

$\varphi^3 = \varphi^2 + \varepsilon = 14 + 4 = 18$

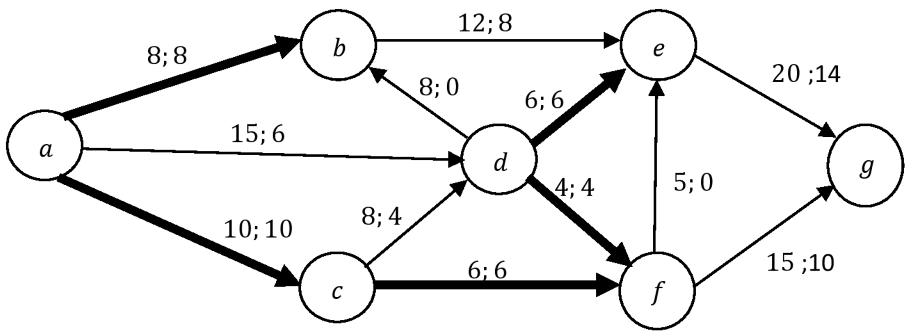


Iteration 4 :

$\mathrm{A} = \{a, d, e, g\}$

$\varepsilon = min\{15 - 0{,}6 - 0{,}20 - 8\} = min\{15{,}6{,}12\} = 6$

$\varphi^4 = \varphi^3 + \varepsilon = 18 + 6 = 24$

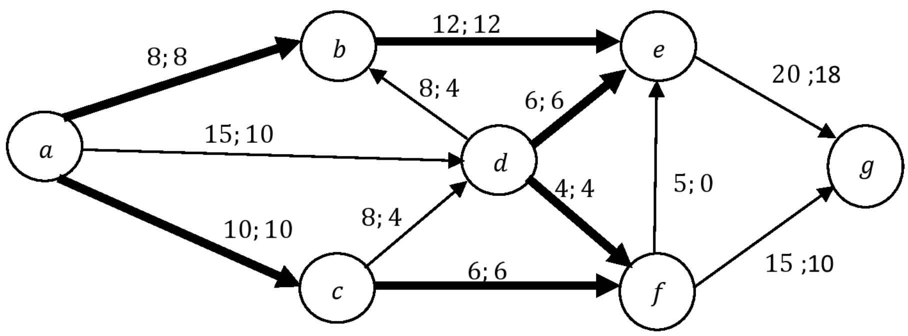


Iteration 5 :

$\mathrm{A} = \{a, d, b, e, g\}$

$\varepsilon = min\{15 - 6{,}8 - 0{,}12 - 8{,}20 - 14\} = min\{9{,}8{,}4{,}6\} = 4$

$\varphi^5 = \varphi^4 + \varepsilon = 24 + 4 = 28$



Iteration 6 :

$A = \{a, d, b, STOP\}$ or $A = \{a, d, c, STOP\}$

We can no longer mark and g is not marked, THEN **finished, the flow is maximum** :

$$\varphi_{max} = \varphi^5 = 28$$

$$\varphi_{max} = \sum \left(\varphi(a,x)/x \in \Gamma_R^+(a)\right) = \varphi^5(a,b) + \varphi^5(a,d) + \varphi^5(a,c) = 8 + 10 + 10 = 28$$

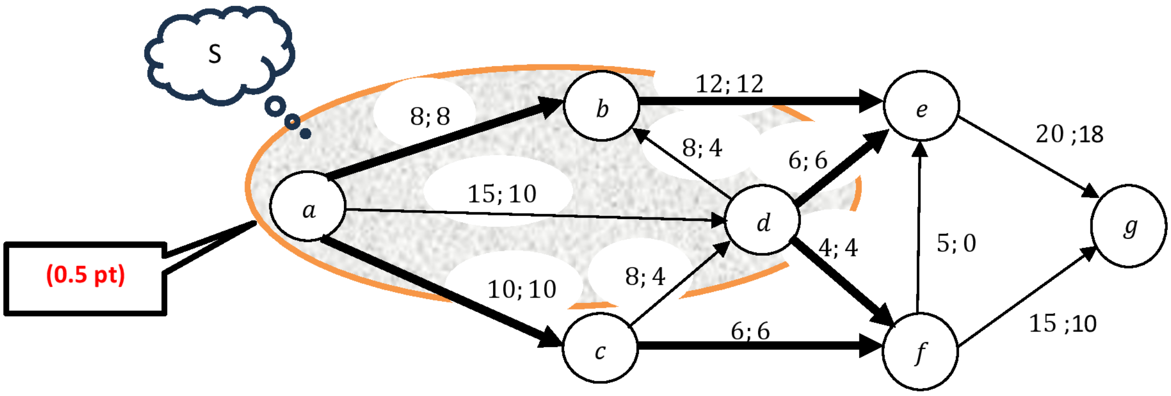
or

$$\varphi_{max} = \sum \left(\varphi(x,g)/x \in \Gamma_R^-(g)\right) = \varphi^5(e,g) + \varphi^5(f,g) = 18 + 10 = 28$$

Then :

$$\boxed{\boldsymbol{\varphi_{max} = 28}}$$

The (a-g) - cut : (3.5 pts)



a) The arcs set of the $(a - g) - cut$ is given by :

$$(a - g) - cut = \{(b,e), (d,e), (d,f), (a,c)\}$$

(0.5 pt)

b) The $(a - g) - cut$ is the partition of vertices :

$$C_p = S \cup P = \{a, b, d\} \cup \{c, e, f, g\}$$

(0.5 pt)

c) The capacity of the $(a - g) - cut$ is equal to :

$$C(C_p) = \sum_{\substack{x \in S \\ y \in P}} c(x,y) = c(b,e) + c(d,e) + c(d,f) + c(a,c) = 12 + 6 + 4 + 10 = \mathbf{32}$$

(0.5 pt)

d) This cut C_P is *not minimal* because : **(0.5 pt)**

- The arcs $(b,e), (d,e), (d,f), (a,c)$ outcoming the cut are saturated.
- The arc (c,d) incoming the cut with a flow $\varphi(c,d) = 4 \neq 0$, so this condition is *not* verified.

e) C_P is *not minimal* So,

$$|\,\varphi_{max}\,| \leq |\,C(C_p)\,|$$

(0.5 pt)

$$28 \leq 32$$

f) The net flow φ_{C_p} crossing the cut C_p is equal to :

$$\left|\varphi_{C_p}\right| = \varphi^{+}_{C_p}(S) - \varphi^{-}_{C_p}(S) = \sum_{\substack{x\in S\\ y\in P}} \varphi(x,y) - \sum_{\substack{x\in P\\ y\in S}} \varphi(x,y)$$

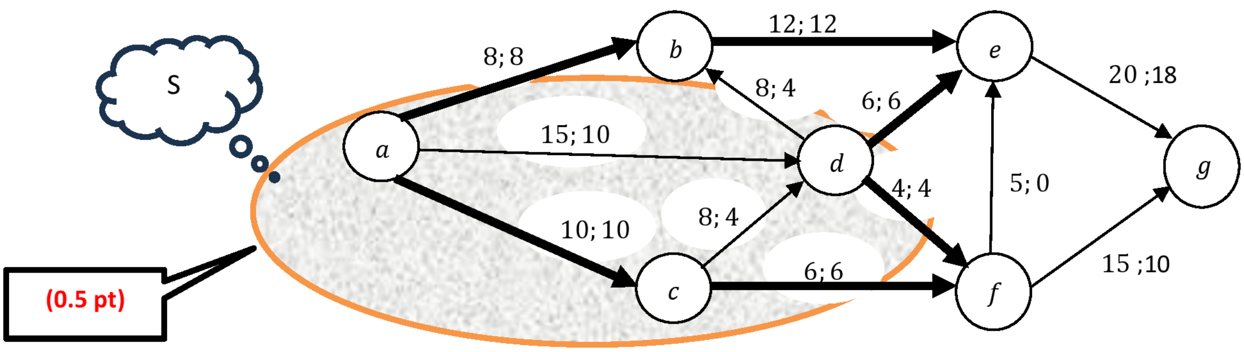
(0.5 pt)

$$\left|\varphi_{C_p}\right| = \big(\varphi(b,e) + \varphi(d,e) + \varphi(d,f) + \varphi(a,c)\big) - \varphi(c,d)$$

$$\left|\varphi_{C_p}\right| = (12 + 6 + 4 + 10) - 4 = 32 - 4 = \mathbf{28}$$

OR

The (a-g) - cut : (3.5 pts)



g) The arcs set of the $(a-g)-cut$ is given by :

$$(a-g)-cut = \{(a,b),(d,b),(d,e),(d,f),(c,f)\}$$

(0.5 pt)

h) The $(a-g)-cut$ is the partition of vertices :

$$C_p = S \cup P = \{a,c,d\} \cup \{b,e,f,g\}$$

(0.5 pt)

i) The capacity of the $(a-g)-cut$ is equal to :

$$C(C_p) = \sum_{\substack{x\in S\\ y\in P}} c(x,y) = c(a,b) + c(d,b) + c(d,e) + c(d,f) + c(c,f) = 8+8+6+4+6 = \mathbf{32}$$

(0.5 pt)

j) This cut C_P is *not minimal* because : **(0.5 pt)**

- The arcs $(a,b),(d,e),(d,f),(c,f)$ outcoming the cut are saturated but the arc (d,b) is not saturated, so this condition is *not* verified.
- No arcs incoming the cut with a flow $\varphi(x,y) = 0$, so this condition is *implicitly* verified.

k) C_P is *not minimal* So,

$$|\varphi_{max}| \leq |C(C_p)|$$

$$28 \leq 32$$

(0.5 pt)

I) The net flow φ_{C_p} crossing the cut C_p is equal to :

$$\left|\varphi_{C_p}\right| = \varphi^{+}_{C_p}(S) - \varphi^{-}_{C_p}(S) = \sum_{\substack{x \in S \\ y \in P}} \varphi(x,y) - \sum_{\substack{x \in P \\ y \in S}} \varphi(x,y)$$

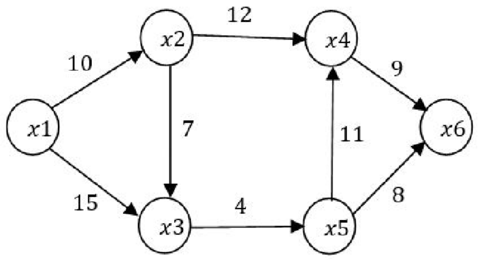
$$\left|\varphi_{C_p}\right| = \big(\varphi(a,b) + \varphi(d,b) + \varphi(d,e) + \varphi(d,f) + \varphi(c,f)\big) - 0$$

$$\left|\varphi_{C_p}\right| = (8+4+6+4+6) - 0 = 28 - 0 = \mathbf{28}$$

(0.5 pt)

Exercise n°=3 : (5 pts)

Let be the following graph G :



Determine a minimal weight path from vertex $\boldsymbol{x1}$ to each of the other vertices of the graph G, indicating the different steps.

		Sommets						
Etapes (*k*)	***D***	**1**	**2**	**3**	**4**	**5**	**6**	
1	$\{x1\}$	0	**<u>10</u>**	15	$+\infty$	$+\infty$	$+\infty$	**1 pt**
2	$\{x1, \boldsymbol{x2}\}$	0	10	**<u>15</u>**	22	$+\infty$	$+\infty$	
3	$\{x1, x2, \boldsymbol{x3}\}$	0	10	15	22	**<u>19</u>**	$+\infty$	**1 pt**
4	$\{x1, x2, x3, \boldsymbol{x5}\}$	0	10	15	**<u>22</u>**	19	27	**1 pt**
5	$\{x1, x2, x3, x5, \boldsymbol{x4}\}$	0	10	15	22	19	**<u>27</u>**	**1 pt**
6	$\{x1, x2, x3, x5, x4, \boldsymbol{x6}\}$	0	10	15	22	19	27	**1 pt**

comprehension questions (MCQ) : (4 pts = 0.25 pt x 16)

Check the correct answer(s) in the following :

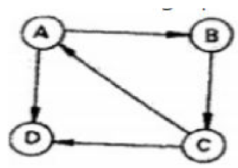
1) **Let X be the adjacency matrix of a graph G with no self loops. The entries along the principal diagonal of X are :**

	all ones
x	all zeros
	both zeros and ones
	different

2) **The in-degree of a vertex in a directed graph is:**

	The number of vertices connected to it
	The number of edges leaving it
x	The number of edges coming into it
	The sum of its neighbors degrees

3) **Consider the graph shown in the figure below :**



Which of the following is a valid strongly component ?

	$\{A, C, D\}$
x	$\{A, C, B\}$
	$\{A, D, B\}$
	$\{B, C, D\}$

4) **A graph is considered complete if :**

	All its edges are collinear
x	All its vertices are adjacent to each other
	It is composed of straight lines
	It is oriented

5) **Chain length is :**

x	The number of edges that compose it
	The number of vertices that compose it
	The number of graphs that compose it
	The number of matrices that compose it

6) **An elementary path can pass through the same arc several times :**

	True
x	False

7) **In a graph G there is one and only one path between every pair of vertices then G is a :**

	Path
	Walk
	Circuit
x	Tree

8) **A graph in which all nodes are of equal degree, is known as :**

	Multigraph
	Non regular graph
x	Regular graph
	Complete graph

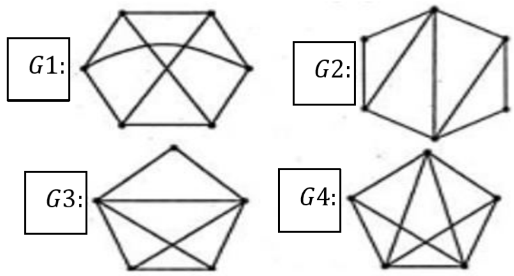
9) **A vertex-edge matrix is composed of elements whose values can be :**

x	0
x	1
	-1
	'True' ou 'False'

10) **The maximum number of edges in a bipartite graph on 12 vertices is :**

	12
	24
x	36
	48

11) **Which one of the following graphs is NOT planar ?**



x	$G1$
	$G2$
	$G3$
	$G4$

12) **Let G be a simple connected planar graph with 13 vertices and 19 edges. Then, the number of faces in the planar embedding of the graph is :**

	6
x	8
	9
	13

13) **If the graph is with triangle, then we apply Euler's property 1 :**

	$m \leq 6 \times n - 3$
x	$m \leq 3 \times n - 6$
	$m \leq 2 \times n - 4$
	$m \leq 4 \times n - 2$

14) **A graph is a tree if and only if :**

	Is planar
	Contains a circuit
x	Is minimally
	Is completely connected

15) **The empty subgraph of a graph contains :**

x	No vertices and no edges
	No vertices but some edges
	Some vertices but no edges
	All the vertices but no edges