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Departement of Mathematics and Computer Science	**1^{st} year Licence Maths**

Probability and descriptive statistics

Exercise 01 (03 points) : Find the median, the first and the third quartile for each the following set of data :

A. 13, 11, 10, 5, 6, 10, 4, 2, 10, 6

$$2, 4, 5, 6, 6, 10, 10, 10, 11, 13 \qquad (0.25\ pts)$$

The median Me_A : we have $n = 10$ an even number so

$$Me_A = \frac{(\frac{n}{2})^{th}\ value + (\frac{n}{2}+1)^{th}\ value}{2} = \frac{6+10}{2} = 8 \qquad (0.25\ pt)$$

The 1^{st} quartile q_{1_A}

$$q_{1_A} = \left(\frac{n}{4}\right)^{th}\ value = 5 \qquad (0.25\ pt)$$

The 3^{rd} quartile q_{3_A}

$$q_{3_A} = \left(\frac{3n}{4}\right)^{th}\ value = 10 \qquad (0.25\ pt)$$

B. 4, 1, 10, 12, 7, 8, 10, 12, 7, 11, 9

$$1, 4, 7, 7, 8, 9, 10, 10, 11, 12, 12 \qquad (0.25\ pt)$$

The median Me_B

$$Me_B = \left(\frac{n+1}{2}\right)^{th}\ value = 9 \qquad (0.25\ pt)$$

The 1^{st} quartile q_{1_B}

$$q_{1_B} = \left(\frac{n}{4}\right)^{th}\ value = 7 \qquad (0.25\ pt)$$

The 3^{rd} quartile q_{3_B}

$$q_{1_B} = \left(\frac{n}{4}\right)^{th}\ value = 11 \qquad (0.25\ pt)$$

Draw the box plot for each set of data A and B. (0.5 pt)

The box plot A has the wider spread for the middle 50% of the data because

$$IQR_A = q_{3_A} - q_{1_A} \quad > \quad IQR_B = q_{3_B} - q_{1_B}$$

Exercise 02 (07 points) : Consider the following frequency table :

Weight (kg) $[e_{i-1}, e_i[$	[30, 40[	[40, 50[	[50, 60[	[60, 70[	$\sum$
Number of packages	10	20	8	2	$n = 40$
$c_i = \frac{e_i + e_{i-1}}{2}$	35	45	55	65	////
$a_i = e_i - e_{i-1}$	10	10	10	10	////
$N_{x=e_i}$	10	30	38	40	////
$n_i \times c_i$	350	900	440	130	1820
$n_i \times c_i^2$	12250	40500	24200	8450	84400

1. The population studied : set of packages (0.25 pt), its size $n = 40$ (0.25 pt),
   The variable studied : weight of package (0.25 pt), its type : Quantitative continuous (0.25 pt).
2. Draw the frequency histogram and the frequency curve. (02 pts)
3. The median,
$$10 \leq 20 < 30 \quad (from\ \ line\ \ N_{x=e_i})$$
$$40 \leq Me < 50$$
   so
$$Me = 40 + (50-40) \times \frac{20-10}{30-10} = 45; \qquad (01\ pt)$$
   The mode :
$$Mo = 40 + (50-40) \times \frac{20-10}{(20-10)+(20-8)} = 44.54 \quad (0.5\ pt)$$
   The rang :
$$R = max - min = 70 - 30 = 40 \qquad (0.25\ pt)$$
   The mean :
$$\overline{x} = \frac{\sum n_i \times c_i}{n} = \frac{1820}{40} = 45.5 \qquad (0.5\ pt)$$
   The variance :
$$var(X) = \frac{\sum n_i \times c_i^2}{n} - \overline{x}^2 = 64.75 \qquad (0.5\ pt)$$
   The standard deviation : $\sigma_X = 8.04 \qquad (0.25\ pt)$
4. We have $P_{[Me,\,\alpha[} = 20\% = \Big(F_\alpha \uparrow - F_{Me} \uparrow\Big) \times 100$
   $\Rightarrow \quad F_{[Me,\,\alpha[} = 0.2 = F_\alpha \uparrow - F_{Me} \uparrow \ \Rightarrow \quad F_\alpha \uparrow = 0.7$ so
$$10 \leq \ N_\alpha \uparrow = F_R \uparrow \times n = 0.7 \times n = 28 \ < 30 \quad \mathbf{(0.25\ pt)}$$
   then :
$$40 \leq \alpha < 50$$
   So : $\alpha = 49$

Exercise 03 (02 points) : Let $(\Omega, \mathcal{F}, P)$ be a probability space. We want to prove that $\mathcal{G} = \{A \in \mathcal{F}, \quad P(A) = 0 \quad ou \quad P(A) = 1\}$ is a tribe on Ω.

Cond.1. $\mathcal{G}$ is no vide because $\Omega \in \mathcal{F}$ and $P(\Omega) = 1$, so $\Omega \in \mathcal{G}$.

Cond.2. Let $A \in \mathcal{G}$ be an event and we show that $\overline{A} \in \mathcal{G}$.

We have

$$\begin{aligned} A \in \mathcal{G} &\Longrightarrow A \in \mathcal{F} \quad and \quad P(A) = 0 \quad or \quad P(A) = 1 \\ &\Longrightarrow \overline{A} \in \mathcal{F} \quad and \quad P(\overline{A}) = 1 - P(A) = 1 - 0 = 1 \quad or \quad P(\overline{A}) = 0 \\ &\Longrightarrow \overline{A} \in \mathcal{F} \quad and \quad P(\overline{A}) = 0 \quad or \quad P(\overline{A}) = 1 \\ &\Longrightarrow \overline{A} \in \mathcal{G}. \end{aligned}$$

Cond.3. Let $(A_i)_{i\geq 0}$ be a sequence of events of $\mathcal{G}$. We want to prove that $\cup_i A_i \in \mathcal{G}$.

$$\begin{aligned} For\ \ all\ \ i \geq 0,\ \ A_i \in \mathcal{G} &\Longrightarrow \forall\, i \geq 0,\ A_i \in \mathcal{F} \quad and \quad P(A_i) = 0 \quad or \quad P(A_i) = 1 \\ &\Longrightarrow \cup_i\ A_i \in \mathcal{F} ....(1)\ becaus\ \ \mathcal{F}\ is\ \ a\ \ tribe \end{aligned}$$

If $P(A_i) = 0$, for all $i \geq 0$, so $P(\cup_i\ A_i) = 0$..........(2)

If **at least** $A_i, i \geq 0$, such that $P(A_i) = 1$, we find $P(\cup_i\ A_i) = 1$..........(3)

So from (1), (2), and (3) we deduce that $\cup_i A_i \in \mathcal{G}$.

Exercise 04 (02 points) : Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let B be an event. Show that P_B is a probability on Ω such as :

$$P_B(A) = \frac{P(A \cap B)}{P(B)} = P(A \mid B).$$

Cond.1. $0 \leq P_B(A) \leq 1$, because
$A \cap B \subseteq B \Rightarrow 0 \leq P(A \cap B) \leq P(B) \Rightarrow 0 \leq \dfrac{P(A \cap B)}{P(B)} \leq 1.$

Cond.2. $P_B(\Omega) = 1$, $because$ $P_B(\Omega) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

Cond.3. Let $(A_i)_{i\geq 0}$ be a sequence of events of $\mathcal{F}$, such as these events are disjoints two by two.

We want to prove that $P_B(\cup_i A_i) = \sum_i\ P_B(A_i)$. We have :

$$\begin{aligned} P_B(\cup_i A_i) &= \frac{P\Big((\cup_i A_i)\ \cap B\Big)}{P(B)} \\ &= \frac{P\Big(\cup_i\ (A_i\ \cap B)\Big)}{P(B)} \end{aligned}$$

Note that the events A_i, $i = 1, ..., n$, are disjoints two by two, then the events $A_i \cap B$ are also disjoints two by two. So :

$$\begin{aligned} P_B(\cup_i A_i) &= \sum_i \frac{P(A_i\ \cap B)}{P(B)} \\ &= \sum_i P_B(A_i) \end{aligned}$$

Exercise 05 (06 points) : A box contains n white balls and 5 black balls.

1. We draw 2 balls randomly. The probability that
   a) A "both balls are white"
   $$P(A) = \frac{card(A)}{card(\Omega)} = \frac{C_n^2}{C_{n+5}^2}$$
   b) B : "both balls are the same color"
   $$P(B) = \frac{card(B)}{card(\Omega)} = \frac{C_n^2 + C_5^2}{C_{n+5}^2}$$
   c) C : "at least one of the two balls is white"
   $$P(C) = \frac{card(C)}{card(\Omega)} = \frac{C_n^1 \times C_5^1 + C_n^2}{C_{n+5}^2}$$
2. We draw 2 balls successively with replacement. What is the probability that
   a) D : "both balls are white"
   $$P(D) = \frac{card(D)}{card(\Omega)} = \frac{n^2}{(n+5)^2}$$
   b) E : "both balls are the same color"
   $$P(E) = \frac{card(E)}{card(\Omega)} = \frac{n^2 + 5^2}{(n+5)^2}$$
   c) F : at least one of the two balls is white"
   $$P(F) = \frac{card(F)}{card(\Omega)} = \frac{5n + n^2}{(n+5)^2}$$