Model Answer + Grading Rubric

Answer to exercise 01 : (PARALLEL MACHINE : 05 Marks)

We have $\lambda = 21$ tasks/s and $\mu = 4$ tasks/s. We have Poisson process for arrivals and exponential service times. We can model this system with an $M/M/m$ queue.

1. The number of processors n in this machine :
   Server utilisation $\varrho = \frac{\lambda}{m\mu} = \frac{21}{4m} = 0.75 \implies m = \frac{21}{0.75\times4} \implies$ $m = 7$ ← 01.50
2. The probability of finding all processors idle :
   We have $P_0 = \left[\frac{(m\rho)^m}{m!(1-\rho)} + \sum_{k=0}^{m-1}\frac{(m\rho)^k}{k!}\right]^{-1}$
   $P_0 = \left[\frac{(7\times0.75)^7}{7!\times0.25} + 1 + (7\times0.75) + \frac{(7\times0.75)^2}{2!} + \frac{(7\times0.75)^3}{3!} + \frac{(7\times0.75)^4}{4!} + \frac{(7\times0.75)^5}{5!} + \frac{(7\times0.75)^6}{6!}\right]^{-1}$
   $P_0 = [87.24580993 + 1 + 5.25 + 13.78125 + 24.1171875 + 31.65380859 + 33.23649902 + 29.08193664]^{-1}$
   $P_0 = [225, 36649168]^{-1}$
   . Hence, $P_0 = 0.00443721$ ← 01.50

We want to lower the utilization rate to 50% (maximum) :

3. Consider x the number of processors (identical to those of the machine) to be added :
   The new server utilisation must be 50% maximum. Hence, $\varrho' = \frac{\lambda}{(m+x)\mu} \leq 0.5 \implies x \geq \frac{\lambda-0.5m\mu}{0.5\mu}$
   $x \geq \frac{21-0.5\times7\times4}{0.5\times4} \implies$ $x \geq 3.5$. We must add at least 4 new processors ← 01.00
4. The average number of running tasks :
   In this case : $\varrho' = \frac{\lambda}{m'\mu} = \frac{21}{11\times4} = 0.477272727$ and $\overline{R} = m\varrho = 11\times0.477272727$. So, $\overline{R} = 5.25$ ← 01.00

Answer to exercise 02 : (QUEUEING SYSTEM : 05 Marks)

We have $\lambda = 5$ requests/minute and $\mu = 7$ requests/minute. The appropriate model for the system is $M/M/1/N$ because :

- The arrival process is Poisson (statement).
- Exponential service time (statement).
- The number of servers is $m = 1$ (statement).
- Service discipline is $FIFO$ (default).
- Finite capacity (N requests in the system/limited memory size).
- Infinite population size (default).

1. The minimal memory space required to reach the objective, if each request needs 2 MB is :
   $\varrho = \frac{\lambda}{\mu} = \frac{5}{7} \approx 0,714285714 \neq 1$, $P_0 = \frac{1-\varrho}{1-\varrho^{c+1}}$, $P_N = \frac{(1-\varrho)\varrho^N}{1-\varrho^{N+1}}$
   The effective request rate accepted by the server is λ_e. To satisfy at least 75% of the received requests, $\frac{\lambda_e}{\lambda} \geq 0.75$ must hold.
   $\frac{\lambda_e}{\lambda} \geq 0.75 \implies \frac{\lambda(1-P_N)}{\lambda} \geq 0.75 \implies 1-P_N \geq 0.75 \implies 1-\frac{(1-\varrho)\varrho^N}{1-\varrho^{N+1}} \geq 0.75 \implies \frac{1-\varrho^N}{1-\varrho^{N+1}} \geq 0.75$
   Hence : $\frac{\lambda_e}{\lambda} \geq 0.75 \implies (1-0.75\varrho)\varrho^N \leq 0.25 \implies \varrho^N \leq \frac{0.25}{1-0.75\varrho} \implies \varrho^N \leq 0,538461538 \implies N\ln(\varrho) \leq \ln(0,538461538)$
   $\frac{\lambda_e}{\lambda} \geq 0.75 \implies N\ln(\varrho) \leq \ln(0,538461538) \implies N \geq \frac{\ln(0,538461538)}{\ln(\varrho)}$ ($\ln(\varrho) < 0$ because $\varrho < 1$)
   Finally, $N \geq 1,839792830$ and at least $N = 2$. Nedded memory = $(N-1)\times2 = 2MB$ ← 01.00
2. The probability that the server becomes idle between the arrival of two requests, knowing that it was unoccupied when the first arrived :
   It may become idle, if he can finish the current request before the next one arrives. On average, a request arrives every 12 secondes. Service time is exponential (Poisson process) of intensity $\mu = 7$ requests/seconde ≈ 0.116666666 requests/seconde :
   $$P(T_n - T_{n-1} < 12) = 1 - e^{-12\mu} = 1 - e^{-12\times0.116666666} \approx 0,7534030360$$ ← 01.50
3. The average number of daily processed requests :
   Since there is a reject and only part of the flow enters the system, we use the effective occupancy rate here :
   $P_N = \frac{(1-\varrho)\varrho^N}{1-\varrho^{N+1}} = 0.229357798$, $P_0 = \frac{1-\varrho}{1-\varrho^{N+1}} = 0.4495412844$,

$\varrho_e = \frac{\lambda_e}{\mu} = \frac{\lambda(1-P_N)}{\mu} = \frac{5\times 0.770642202}{7} = 0.5504587157$

Duration of activity per day (24 hours) : $24 \times \varrho_e = 24 \times 0.5504587157 = 13.2110091768$ hours $= 792,660550608$ minutes.

$Nb = \lfloor \frac{\text{Activity duration}}{\overline{S}} \rfloor = \lfloor \text{Activity duration} \times \mu \rfloor = \lfloor 792,660550608 \times 7 \rfloor = \lfloor 5548,623854256 \rfloor \implies \boxed{Nb = 5548 \text{ requests}}$ 01.00

4. The average number of waiting requests :

$\overline{Q} = \frac{(m\varrho)^m \varrho P_0}{m!(1-\varrho)^2}\left[1 - \varrho^{N-m} - (N-m)(1-\varrho)\varrho^{N-m}\right]$

$\overline{Q} = \frac{(1\times 0.714285714)^1 0.714285714 \times 0.4495412844}{1!(1-0.714285714)^2}\left[1 - 0.714285714^{2-1} - (2-1)(1-0.714285714)0.714285714^{2-1}\right]$

Hence : $\boxed{\overline{Q} = 0.22935779816513768 \text{ requests}}$ 01.50

Answer to exercise 03 : (Queueing network : 10 Marks)

Nous avons : $\gamma = 5, m_1 = +\infty, m_2 = 2, m_3 = 1, \mu_1 = 1, \mu_2 = 5, \mu_3 = 10.$

1. Internal routing probability matrix : $\boxed{\begin{pmatrix} 0 & 1 & 0 \\ \alpha & 0 & 1-\alpha \\ \alpha & 0 & 0 \end{pmatrix}}$ 01.00, External routing probability matrix : $\boxed{\begin{pmatrix} 0 \\ 0 \\ 1-\alpha \end{pmatrix}}$ 01.00

2. Effective arrival rates λ_i : $\begin{cases} \lambda_1 = \gamma + \alpha\lambda_2 + \alpha\lambda_3 \\ \lambda_2 = \lambda_1 \\ \lambda_3 = (1-\alpha)\lambda_2 \end{cases} \Rightarrow \begin{cases} (1-\alpha)\lambda_1 = 5 + \alpha\lambda_3 \\ \lambda_2 = \lambda_1 \\ \lambda_3 = (1-\alpha)\lambda_2 \end{cases}$

$\Rightarrow \begin{cases} (1-\alpha)\lambda_1 = 5 + \alpha(1-\alpha)\lambda_1 \\ \lambda_2 = \lambda_1 \\ \lambda_3 = (1-\alpha)\lambda_1 \end{cases} \Rightarrow \begin{cases} (1-\alpha)^2\lambda_1 = 5 \\ \lambda_2 = \lambda_1 \\ \lambda_3 = (1-\alpha)\lambda_1 \end{cases} \Rightarrow \boxed{\begin{cases} \lambda_1 = \frac{5}{(1-\alpha)^2} \\ \lambda_2 = \frac{5}{(1-\alpha)^2} \\ \lambda_3 = \frac{5}{1-\alpha} \end{cases}}$ 01.50

Note that $\alpha = 1$ cannot be a solution.

3. Values of α which ensure the stability of the network :

$\begin{cases} FA_1 \text{ Stable} \\ FA_2 \text{ Stable} \\ FA_3 \text{ Stable} \end{cases} \Rightarrow \begin{cases} \varrho_1 = 0 < 1 \\ \varrho_2 = \frac{\lambda_2}{m_2\mu_2} = \frac{5}{2\times 5\times(1-\alpha)^2} < 1 \\ \varrho_3 = \frac{\lambda_1}{m_1\mu_1} = \frac{5}{1\times 10\times(1-\alpha)} < 1 \end{cases} \Rightarrow \begin{cases} 2(1-\alpha)^2 > 1 \\ 2(1-\alpha) > 1 \end{cases} \Rightarrow \begin{cases} \alpha^2 - 2\alpha + \frac{1}{2} > 0 \\ \alpha < \frac{1}{2} \end{cases}$

Solutions of the quadratic equation $\alpha^2 - 2\alpha + \frac{1}{2} = 0$ are :

$\alpha = 1 \pm \frac{\sqrt{2}}{2} \implies \alpha_1 = 1.70710678 \quad \alpha_2 = 0.29289321$

network stable $\implies \begin{cases} 0 \le \alpha < 0.29289321 \\ 0 \le \alpha < 0.5 \end{cases}$

We conclude that, the network to be stable, α must verify the following condition :

$\boxed{0 \le \alpha < 0.29289321}$ or $\boxed{\alpha \in [0, 0.29289321[}$ 01.00

For $\alpha = \frac{1}{5}$:

$\begin{cases} \lambda_1 = \frac{5}{(1-\alpha)^2} = \frac{5}{(\frac{4}{5})^2} = \frac{125}{16} \\ \lambda_2 = \frac{5}{1-\alpha} = \frac{5}{\frac{4}{5}} = \frac{25}{4} \\ \lambda_3 = \frac{4\alpha}{0.6-\alpha^2} = \frac{4\times 0.3}{0.6-0.3^2} \end{cases} \Rightarrow \boxed{\begin{cases} \lambda_1 = 7.8125 \\ \lambda_2 = 7.8125 \\ \lambda_3 = 6.25 \end{cases}} \begin{cases} \varrho_1 = 0 \\ \varrho_2 = \frac{\lambda_2}{m_2\mu_2} = \frac{7.8125}{2\times 5} \\ \varrho_3 = \frac{\lambda_3}{m_3\mu_3} = \frac{6.25}{1\times 10} \end{cases} \Rightarrow \boxed{\begin{cases} \varrho_1 = 0 \\ \varrho_2 = 0.78125 \\ \varrho_3 = 0.625 \end{cases}}$

Queue 1 : of type $M/M/+\infty$. Queue 2 : de type $M/M/2$. Queue 3 : de type $M/M/1$.

4. The average number of customers waiting in each queue and in the network :

(a) $\boxed{\overline{Q_1} = 0}$ 00.25

(b) $P_0(2) = \left[\frac{(m_2\rho_2)^{m_2}}{m_2!(1-\rho_2)} + \sum_{k=0}^{m_2-1}\frac{(m_2\rho_2)^k}{k!}\right]^{-1} = \left[\frac{(2\times 0.78125)^2}{2!(1-0.78125)} + 1 + 2\times 0.78125\right]^{-1}$

$P_0(2) = [5.58035714 + 1 + 1.5625]^{-1} \Rightarrow \boxed{P_0(2) = 0.12280701}$ 00.25

$\zeta_2 = \frac{(m_2\rho_2)^{m_2}}{m_2!(1-\rho_2)} P_0(2) = \frac{(2\times 0.78125)^2}{2!(1-0.78125)} P_0(2) = 5.58035714 \times 0.12280701 \Rightarrow \boxed{\zeta_2 = 0.68530697}$ 00.25

$\overline{Q_2} = \frac{\zeta_2 \rho_2}{1-\rho_2} = \frac{0.68530697\times 0.78125}{1-0.78125} \Rightarrow \boxed{\overline{Q_2} = 2.44752506}$ 00.25

(c) $\overline{Q_3} = \overline{N_3} - \overline{R_3} = \frac{\varrho_1}{1-\varrho_1} - m_1\varrho_1 = \frac{0.625}{1-0.625} - 0.625 = 1.66666666 - 0.625 \Rightarrow \boxed{\overline{Q_3} = 1.04166666}$ 00.25

(d) $\overline{Q_R} = \sum_{i=1}^{3} \overline{Q_i} = 0 + 2.44752506 + 1.04166666 \Rightarrow \boxed{\overline{Q_R} = 3.48919172}$ 00.25

5. The average number of customers in each queue and in the network :

(a) $\overline{N_1} = \overline{R_1} = \frac{\lambda_1}{\mu_1} = \frac{7.8125}{1} \Rightarrow \boxed{\overline{N_1} = 7.8125}$ 00.25

(b) $\overline{N_2} = \overline{Q_2} + \overline{R_2} = 2.44752506 + 2 \times 0.78125 \Rightarrow \boxed{\overline{N_2} = 4.01002506}$ 00.25

(c) $\overline{N_3} = \frac{\varrho_3}{1-\varrho_3} = \frac{0.625}{1-0.625} \Rightarrow \boxed{\overline{N_3} = 1.66666666}$ 00.25

(d) $\overline{N_R} = \sum_{i=1}^{3} \overline{N_i} = 7.8125 + 4.01002506 + 1.66666666 \Rightarrow \boxed{\overline{N_R} = 13.48919172}$ 00.25

6. The average residence time in each queue and in the network :

(a) $\overline{T_1} = \frac{\overline{N_1}}{\lambda_1} = \frac{7.8125}{7.8125} \Rightarrow \boxed{\overline{T_1} = 1}$ 00.25

(b) $\overline{T_2} = \frac{\overline{N_2}}{\lambda_2} = \frac{4.01002506}{7.8125} \Rightarrow \boxed{\overline{T_2} = 0.51328320}$ 00.25

(c) $\overline{T_3} = \frac{\overline{N_3}}{\lambda_3} = \frac{1.66666666}{0.625} \Rightarrow \boxed{\overline{T_3} = 2,66666666}$ 00.25

(d) $\overline{T_R} = \frac{\overline{N_R}}{\lambda_R} = \frac{\overline{N_R}}{\sum_{i=1}^{2} \gamma_i} = \frac{13.48919172}{5} \Rightarrow \boxed{\overline{T_R} = 2,69783834}$ 00.25

7. The average waiting time in each queue and in the network :

(a) $\boxed{\overline{W_1} = 0}$ 00.25

(b) $\overline{W_2} = \frac{\overline{Q_2}}{\lambda_2} = \frac{2.44752506}{7.8125} \Rightarrow \boxed{\overline{W_2} = 0.31328320}$ 00.25

(c) $\overline{W_3} = \frac{\overline{Q_3}}{\lambda_3} = \frac{1.04166666}{0.625} \Rightarrow \boxed{\overline{W_3} = 0.166666666}$ 00.25

(d) $\overline{W_R} = \frac{\overline{Q_R}}{\lambda_R} = \frac{\overline{Q_R}}{\sum_{i=1}^{2} \gamma_i} = \frac{3.48919172}{5} \Rightarrow \boxed{\overline{W_R} = 0.69783834}$ 00.25

8. The probability that the network is not empty :
$Pr(\text{Network not empty}) = 1 - Pr(\text{Network empty}) = 1 - (Pr(\text{Queue}_1 \text{ empty and Queue}_2 \text{ empty and Queue}_3 \text{ empty}))$
$= (1 - P_0(\text{Queue}_1) \times P_0(\text{Queue}_2) \times P_0(\text{Queue}_3))$
$Pr(\text{Réseau non vide}) = 1 - e^{\frac{-\lambda_1}{\mu_1}} \times P_0(2) \times P_0(3) = 1 - e^{\frac{-7.8125}{1}} \times P_0(2) \times (1 - \rho_3)$
$= 1 - 0.00040464 \times 0.12280701 \times 0.375 = 1 - 0.00001863$
Hence : $\boxed{Pr(\text{Network not empty}) = 0.99998137}$ 01.00