Correction of exam (Math 2).

Exercise 1.

1) Find:

$$A^t = \begin{pmatrix} 0 & -3 & -1 \\ 1 & 4 & 1 \\ -1 & -3 & 0 \end{pmatrix} \ (0,5), \quad A-3B = \begin{pmatrix} -3 & -2 & 2 \\ -3 & 1 & -6 \\ -7 & 1 & 0 \end{pmatrix} \ (0,5), \quad B \in \mathcal{M}_{3,3}, \ C \in \mathcal{M}_{3,1} \ (0,5)$$

CB: Is not defined because $1 \neq 3$ (0,5)

$$BC = \underbrace{\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix}}_{\in \mathcal{M}_{3,3}} \underbrace{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}_{\in \mathcal{M}_{3,1}} = O \in \mathcal{M}_{3,1} = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix} \ (0,5).$$

$$\det(B) = 2 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 4 \ (0,5)$$

2) prove that: $A^2 - 3A + 2I_3 = 0$.

$$A^2 = A.A = \begin{pmatrix} -2 & 3 & -3 \\ -9 & 10 & -9 \\ -3 & 3 & -2 \end{pmatrix} \ (1), \quad 3A = \begin{pmatrix} 0 & 3 & -3 \\ -9 & 12 & -9 \\ -3 & 3 & 0 \end{pmatrix} \ (0,5), \quad 2I = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \ (0,5)$$

Exercise 2: (6/6)

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x^2 + 1 & \text{if } x > 0 \end{cases}$$

Is $x \in]-\infty, 0[$ then $f(x) = \dfrac{\sin(x)}{x}$ is defined and differentiable on $]-\infty, 0[$ and $f'(x) = \dfrac{x\cos(x) - \sin(x)}{x^2}$ (0,5)

Is $x = 0$ then $f(x) = 1 \Rightarrow f'(x) = 0$ (0,5)

Is $x \in]0, +\infty[$: $f(x) = x^2 + 1$ is defined and differentia[ble]
[o]n $]0, +\infty[$ and $f'(x) = 2x$ (0,5)

[Th]en we show if f is differentiable at $x_0 = 0$.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \overset{>}{\to} 0} \frac{(x^2+1) - 1}{x - 0} = \lim_{x \overset{>}{\to} 0} \frac{x^2}{x} = 0. \ (1)$$

$$\lim_{x \overset{<}{\to} x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \overset{<}{\to} 0} \frac{\frac{\sin(x)}{x}-1}{x-0} = \lim_{x \overset{<}{\to} 0} \frac{\sin(x)-x}{x^2} = \frac{0}{0} \text{ F.I}$$

$$= \lim_{x \overset{<}{\to} 0} \frac{\cos(x)-1}{2x} = \frac{0}{0} = \lim_{x \overset{<}{\to} 0} \frac{-\sin(x)}{2} = 0 \quad 1$$

then we have f is differentiable on $\mathbb{R}$ 0.m

* Find $\lim_{x\to 0} \frac{x\cos(x)-\sin(x)}{x^3} = \frac{0}{0}$.

$$= \lim_{x\to 0} \frac{\cos(x) - x\sin(x) - \cos(x)}{x^3} = \lim_{x\to 0} \frac{-x\sin(x)}{3x^2} = \lim_{x\to 0} \frac{-\sin(x)}{3x} \quad 2.$$

$$= \lim_{x\to 0} \frac{-1\cos(x)}{3} = -1/3.$$

Exercice 3: 4/4

* prove that: $g(x) = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$.

$a=-1, \quad b=1, \quad c=1$

$$g(x) = \frac{-1}{x} + \frac{1}{x-1} + \frac{1}{(x-1)^2} \quad 2$$

* Calculate $I = \int g(x)\,dx$.

$$I = \int \left(\frac{-1}{x} + \frac{1}{x-1} + \frac{1}{(x-1)^2}\right) dx.$$

$$I = \int \frac{-1}{x}\,dx + \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx \quad 2.5$$

$$I = -\ln|x| + \ln|x-1| + \frac{(-1)}{(2-1)(x-1)} + C$$

$$I = \ln\left|\frac{x-1}{x}\right| - \frac{1}{(x-1)} + C.$$

Exercise 3: 5/5

* the type of differential equation is : linear of order 1. 1

* Solve the problem (P).

we have : $y'\sin(x) - y\cos(x) = 3x^2\sin^2(x)$ (*)

first we solve. $y'\sin(x) - y\cos(x) = 0$.

$\Rightarrow y' = \frac{\cos(x)}{\sin(x)} y \Rightarrow \frac{dy}{y} = \frac{\cos(x)}{\sin(x)} dx$.

$\Rightarrow \int \frac{dy}{y} = \int \frac{\cos(x)}{\sin(x)} dx \Rightarrow \ln|y| = \ln|\sin(x)| + C$.

2

$\Rightarrow \boxed{y = k\sin(x)}$. we put y in the DE (*)

$y = k(x)\sin(x) \Rightarrow y' = k'(x)\sin(x) + k(x)\cos(x)$.

$(k'(x)\sin(x) + k(x)\cos(x))\sin(x) - k(x)\sin(x)\cos(x) = 3x^2\sin^2(x)$

$k'(x)\sin^2(x) = 3x^2\sin^2(x) \Rightarrow k'(x) = 3x^2 \Rightarrow \boxed{k(x) = x^3 + C}$

then we have the general solution is

$y = (x^3 + C)\sin(x)$ 1.5

If $y(\frac{\pi}{2}) = 0 \Rightarrow ((\frac{\pi}{2})^3 + C)\sin(\frac{\pi}{2}) = 0 \Rightarrow$

$\Rightarrow C = -(\frac{\pi}{2})^3$.

$\Rightarrow \boxed{y = \left(x^3 + (\frac{\pi}{2})^3\right)\sin(x)}$. 0.5

$y = \left(x^3 - \frac{\pi^3}{8}\right)\sin(x)$.