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Master Mathematics
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Correction————

Exercise 01 3

Theorem 1 *Suppose $f, g, h \in C^1$, x^* is a local minimizer of f, then $\exists \lambda \in \mathbb{R}^m$; $\mu \in \mathbb{R}^p$ such that*

$$\begin{aligned} \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla g_j(x^*) &= 0 \\ h(x^*) &= 0 \\ g(x^*) &\leq 0 \\ \forall j &\in \{1, 2, ..., p\}, \quad \mu_j \geq 0 \\ \forall j &\in \{1, 2, ..., p\}, \quad \mu_j g_j(x^*) = 0 \end{aligned}$$

Exercise 02

Consider the following problem

$$\begin{cases} f(x, y) = x(1 + \frac{\pi}{2}) + 2y \\ st \\ g(x, y) = 2 - xy - \frac{\pi}{8}x^2 \end{cases}$$

Qualification

01 Gradient is $\begin{pmatrix} -y - \frac{1}{4}\pi x \\ -x \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Lagrange Function

0.5 $$L(x, y, \lambda) = f(x, y) + \lambda g(x, y) = x(1 + \frac{\pi}{2}) + 2y + \lambda \left(2 - xy - \frac{\pi}{8}x^2\right)$$

Critical point

Gradient of L is

0.5+01 $$\begin{pmatrix} 1 + \frac{1}{2}\pi + y\lambda + \frac{1}{4}\pi x\lambda \\ 2 + x\lambda \\ 2 - xy - \frac{\pi}{8}x^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \lambda x = -2 \\ \lambda y = -1 \end{cases}$$

02 $\lambda \neq 0 \Rightarrow \lambda = \pm\sqrt{\frac{\pi+4}{4}}$ and the critical point is $(x, y) = \left(4\sqrt{\frac{1}{\pi+4}}, 2\sqrt{\frac{1}{\pi+4}}\right)$

Conclusion: by natutre critical point, we can find that $(x,y) = \left(4\sqrt{\frac{1}{\pi+4}}, 2\sqrt{\frac{1}{\pi+4}}\right) = \min f(x,y)$.

Exercise 03 02

Consider the problem

$$\left\{\begin{array}{c} \min f(x,y) = 2x^2 + 2xy + y^2 - 10x - 10y \\ st \\ g_1(x,y) = x^2 + y^2 - 5 \leq 0 \\ g_2(x,y) = 6x + 2y - 12 \leq 0 \\ (x,y) \in \mathbb{R}^2 \end{array}\right. \tag{1}$$

- *The existence and uniqueness of the solution*

f is continuous and f is coercive because it is a quadratic function whose associated matrix

$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is defined positive,

02 • The domain of constraints is closed because its complement is open. So the problem (1) admits at least one solution.

01 • f is strictly convex because it is a quadratic function of positive definite associated matrix

The domain of feasible solutions is convex because g_1 and g_2 are convex.

Consequently, the problem (1) admits a unique solution

- *Active point*

01 $g_i(x) \leq 0$ is active in $x^* \in \Omega$ if $g_i(x^*) = 0$

- *Example*

01 The point $(1,2)$ is active g_1 and inactive in g_2

- *Solve the problem (1)*

The problem is optimization of objective function under linear constraints inequalities so we use KKT method to solve this problem.

Qualification

01 Qualified (Linear)

Lagrange Function

$L(x,y,\lambda,\mu) = 2x^2 + 2xy + y^2 - 10x - 10y + \lambda\left(x^2 + y^2 - 5\right) + \mu\left(6x + 2y - 12\right)$

Critical Point

01 $\left\{\begin{array}{l} \nabla L(x,y,\lambda,\mu) = 0 \\ \lambda\left(x^2 + y^2 - 5\right) = 0 \\ \mu\left(6x + 2y - 12\right) = 0 \end{array}\right.$

We have 4 possiblities

Then the point $(1,2)$ is a critical point of $f(x,y)$.

So,

03 $$(1,2) = \min_{\Omega} f(x,y)$$