L'arbi Ben M'hidi University

Faculty: Exact Sciences, Natural and Life sciences **In:** 14/05/2024
Department : Mathematics and Computer Science. **Duration:** 1h30min
Level: Second year of Bachelor Mathematics

Algebra 4 exam

Exercise 1: (6 *pts*)

I/ *Show that :*
1/ $\forall A, B \in S_n(\mathbb{R}), A.B \in S_n(\mathbb{R}) \Longleftrightarrow A.B = B.A.$ (2 *pts*)
2/ $\forall A \in S_n(\mathbb{R}) \cap GL_n(\mathbb{R}), A^{-1} \in S_n(\mathbb{R})$ (2 *pt*)
II/ Let E *be a vector space on* $\mathbb{R}$, F *be a vector subspace of* E *and* $\beta = \{v_1, .., v_p\}$ *a basis of* F, *and let* $b : E \times E \longmapsto \mathbb{R}$ *be a bilinear form*
Show that if $w \in E$ *is orthogonal to any element of* β *then* w *is orthogonal to any* x *de* F (2 *pts*)

Exercise 2: (6*pts*)

Let $b : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ *bea symmetric bilinear form such as:*

$$\forall (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : b(x, y) = 2x_1y_1 + x_2y_2 + 5x_3y_3 - x_1y_2 - x_2y_1 - x_1y_3 - x_3y_1 - x_2y_3 - x_3y_2$$

1. *Determine the quadratic form associated to* b.
2. *Give a square reduction of Gauss for the quadratic form* q *and determine* $sign(q)$ *and* $rg(q)$.
3. *Is* q *definied positive? Justify your answer.*
4. *Determine the set of isotropic vectors* $I(q)$.

Exercise 3:(8 pts)

Let M *be the following matrix:*

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

1. *Determine the bilinear form* b *associated to the matrix* M *in the canonical basis of* $\mathbb{R}^3$.
2. *Let* $\beta' = \{v_1(1, 2, 1), v_2(-1, 2, 0), v_3(1, 0, 1)\}$.
   (a) *Show that* β' *is a basis of* $\mathbb{R}^3$.
   (b) *Find the matrix* M' *associated to the bilinear form* b *in the basis* β' *using the matrix* $P_{\beta_c \to \beta'}$.
   (c) *Is* β' *an orthonormal basis? Justify your answer.*

Correction

Exercise 1:
I/
1/ $\forall A, B \in S_n(\mathbb{R})$:

$$\begin{array}{rcl} \text{a/ } A.B & \in & S_n(\mathbb{R}) \Longrightarrow A.B = B.A? \ (1pt) \\ A.B & \in & S_n(\mathbb{R}) \Longrightarrow (A.B)^t = A.B \ (\text{mais on sait que } (A.B)^t = B^t.A^t) \\ & \Longleftrightarrow & B^t.A^t = A.B (\text{but } A, B \in S_n(\mathbb{R}) \text{ it means that } B^t = B \text{ and } A^t = A) \\ & \Longleftrightarrow & B.A = A.B \\ \text{b/ } A.B & = & B.A \Longrightarrow A.B \in S_n(\mathbb{R})? \ (1pt) \\ A.B & = & B.A \Longleftrightarrow (A.B)^t = (B.A)^t \\ & \Longrightarrow & (A.B)^t = A^t.B^t (\text{but } A, B \in S_n(\mathbb{R}) \text{ it means that } B^t = B \text{ and } A^t = A) \\ & \Longrightarrow & (A.B)^t = A.B \Longrightarrow A.B \in S_n(\mathbb{R}) \end{array}$$

2/ Let $A \in S_n(\mathbb{R}) \cap GL_n(\mathbb{R})$; $(A^{-1})^t = (A^t)^{-1}$ but A is symmetric then $(A^{-1})^t = (A^t)^{-1} = A^{-1}$ which gives that $A^{-1} \in S_n(\mathbb{R})$. $(1pt)$
II/We have $\beta = \{v_1, .., v_p\}$ *is abasis of* F; Let $x \in F$ then $\exists \alpha_1, \alpha_2, ..., \alpha_p \in \mathbb{R} : x = \sum_{i=1}^{p} \alpha_i v_i$.. (0.5)
So $b(w, x) = b\left(w, \sum_{i=1}^{p} \alpha_i v_i\right) = \sum_{i=1}^{p} \alpha_i b(w, v_i)$ because b is a bilinear form (0.75). But w is orthogonal to any element of F hence $\forall i = \overline{1;p} : b(w, v_i) = 0$ wich gives $b(w, x) = 0$ tnen w is orthogonal to any element x of F. (0.75)

Exercise 2:
We ahve:

$$\forall (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : b(x, y) = 2x_1y_1 + x_2y_2 + 5x_3y_3 - x_1y_2 - x_2y_1 - x_1y_3 - x_3y_1 - x_2y_3 - x_3y_2$$

1. *The quadratic form associated to* b. $(1pt)$
   $q(x) = b(x, x) = 2x_1^2 + x_2^2 + 5x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$

2. The *square reduction of Gauss for the quadratic form* q *(2pts)*

$$\begin{aligned}
q(x) &= 2x_1^2 + x_2^2 + 5x_3^2 + -2x_1x_2 - 2x_1x_3 - 2x_2x_3 \\
&= 2x_1^2 + 2(-x_2 - x_3)x_1 + x_2^2 + 5x_3^2 - 2x_2x_3 \\
&= 2\left[x_1^2 + 2\left(\frac{-x_2 - x_3}{2}\right)x_1\right] + x_2^2 + 5x_3^2 - 2x_2x_3 \\
&= 2\left(x_1 - \frac{1}{2}(x_2 + x_3)\right)^2 - 2\left(\frac{x_2 + x_3}{2}\right)^2 + x_2^2 + 5x_3^2 - 2x_2x_3 \\
&= 2\left(x_1 - \frac{1}{2}(x_2 + x_3)\right)^2 + \frac{1}{2}x_2^2 + \frac{9}{2}x_3^2 - 3x_2x_3 \\
&= 2\left(x_1 - \frac{1}{2}(x_2 + x_3)\right)^2 + \frac{1}{2}x_2^2 + 2\left(-\frac{3}{2}x_3\right)x_2 + \frac{9}{2}x_3^2 \\
&= 2\left(x_1 - \frac{1}{2}(x_2 + x_3)\right)^2 + \frac{1}{2}\left(x_2^2 + 2(-3x_3)x_2\right) + \frac{9}{2}x_3^2 \\
&= 2\left(x_1 - \frac{1}{2}(x_2 + x_3)\right)^2 + \frac{1}{2}(x_2 - 3x_3)^2 - \frac{1}{2}(-3x_3)^2 + \frac{9}{2}x_3^2 \\
&= 2\left(x_1 - \frac{1}{2}(x_2 + x_3)\right)^2 + \frac{1}{2}(x_2 - 3x_3)^2
\end{aligned}$$

$sign(q) = (p, n) = (2, 0)\,..\,(0.5)$ *and* $rg(q) = p + n = 2..\,(0.5)$.

3. q *is defined if and only if* $q(x) = 0 \Longleftrightarrow x = 0$ *and is positive if and only if* $q(x) \geq 0$

   By the *square reduction of Gauss* $q(x)$ *is a sum of two squares therefore* $q(x) \geq 0$ (0.5)

   Now for $x = (2, 3, 1)$ we find $q(x) = 0$ then q is not defined. (0.5)

4. *The set of isotropic vectors* $I(q)$. *(1pt)*

   We have $I(q) = \{x \in \mathbb{R}^3, q(x) = 0\}$

$$\begin{aligned}
q(x) &= 0 \Leftrightarrow \begin{cases} x_1 - \frac{1}{2}(x_2 + x_3) = 0 \\ x_2 - 3x_3 = 0 \end{cases} \\
&\Leftrightarrow \begin{cases} x_1 = 2x_3 \\ x_2 = 3x_3 \end{cases}
\end{aligned}$$

   $I(q) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 / x_1 = 2x_3 \text{ and } x_2 = 3x_3\}$

Exercise 3:(8 pts)

1. *the bilinear form* b *associated to the matrix* M *in the canonical basis of* $\mathbb{R}^3$ is:

$$\begin{aligned}
b(x, y) &= X^t MY......(0.5) \\
&= x_1y_1 - x_1y_2 + 2x_1y_3 + x_2y_1 + 2x_3y_1 + x_3y_2 + x_3y_3...(1.5)
\end{aligned}$$

2. *Let* $\beta' = \{v_1(1, 2, 1), v_2(-1, 2, 0), v_3(1, 0, 1)\}$.

(a) Let us show that β' *is a basis of* $\mathbb{R}^3$.

Put $P = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $\det P = 2 \neq 0$ then $v_1\,(1,2,1)\,, v_2\,(-1,2,0)$ and $v_3\,(1,0,1)$ are linearly independent $(1pt)$, and since $\dim \beta' = \dim \mathbb{R}^3 = 3$ then β' is a basis of $\mathbb{R}^3\,(1pt)$

(b) *Find the matrix M' associated with the bilinear form b in the basis β' using the matrix $P_{\beta_c \to \beta'}$.*

We have : $M' = P^t M P ...\,(1pt)$

$$\begin{aligned} M' &= \begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ M' &= \begin{pmatrix} 10 & -5 & 10 \\ 3 & 1 & 1 \\ 6 & -3 & 6 \end{pmatrix} ...\,(1pt) \end{aligned}$$

(c) β' *is an orthonormal basis* $\Longleftrightarrow \forall i, j \in \{1,2,3\}\,, b\,(v_i, v_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$ ... $(1pt)$

we have $b\,(v_1, v_2) = -5 \neq 0$ so β' *is not an orthonormal basis*.....$(1pt)$