Université L'arbi Ben Mhidi OEB
Département de séances de la matière
Module : Relativité Restreinte
Année universitaire 2023/2024

Durée : 1h30

Réponses

1. **Transformations de Lorentz, vitesse :**

(a) Les transformations des coordonnées (**ct**,**x**,**y**,**z**) : (2pts)

$$\begin{cases} x = \gamma\ (\ x' + B\ c\ t) \\ y = y'\ , z = z' \\ c\,t = \gamma\ (c\,t' + B\ x') \end{cases} \qquad B = \frac{V}{C} \quad , \quad v = v\ e_x \tag{1}$$

(b) Les transformations de composantes de vitesse :
$\vec{v} = \frac{d\vec{r}}{dt}$, $\vec{v'} = \frac{d\vec{r'}}{dt'}$ (1pts)
De (1) : (3 x 0.5pts)

$$\begin{cases} dx' = \gamma\ (\ dx' - V\ dt) \\ dy' = dy\ , dz' = dz \\ dt' = \gamma\ (dt - \frac{V}{c^2}dx)) \end{cases} \Rightarrow V'_x = \frac{dx - V dt}{dt - \frac{V}{c^2}} \ , \ V'_y = \frac{dy}{\gamma(dt - \frac{V}{c^2}dx)} \ , \ V'_z = \frac{dz}{\gamma(dt - \frac{V}{c^2}dx)}$$

$\Rightarrow V'_x = \frac{Vx - V}{1 - \frac{V}{c^2}dx}$, $V'_y = \frac{Vy}{\gamma(1 - \frac{V}{c^2}dx)}$, $V'_z = \frac{Vz}{\gamma(1 - \frac{V}{c^2}dx)}$

(c) Les cas limites intéressants $(v \ll c) \Rightarrow c \to \infty$:
$V'_x = V_x - V$, $V'_y = V_y$ $V'_z = V_z$ (Lois de composition de vitesse) (0.5pts)

(d) $ds^2 = c^2\ dt^2 - dx^2 - dy^2 - dz^2$ (0.5pts)

$ds^2 > 0 \rightarrow$ intervalle genre temps (3 x 0.5pts)
$ds^2 < 0 \rightarrow$ intervalle genre espace
$ds^2 = 0 \rightarrow$ intervalle genre lumiere

(e) Quadri-vitesse $\overline{\mathbf{U}}$: $\overline{\mathbf{U}} = \frac{d\,\overline{R}}{d\,\tau}$ $\overline{R}$: vecteur position. τ : le temps propre.

$$\begin{cases} \overline{U}^\alpha = \frac{dx^\alpha}{d\tau} \quad , \quad \alpha = 0, 1, 2, 3 \\ \tau = \frac{t}{\gamma_p} \Rightarrow d\tau = \frac{dt}{\gamma_p}\gamma_p = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \end{cases}$$ (2 x 0,5pts)

$$\begin{cases} V^0 = \frac{dx^0}{d\tau} = \gamma_p \frac{c\,dt}{dt} = \gamma_p c \\ V^1 = \frac{dx^1}{d\tau} = \gamma_p \frac{dx^1}{dt} = \gamma_p \frac{dx}{dt} = \gamma_p V_x \\ V^2 = \frac{dx^2}{d\tau} = \gamma_p \frac{dx^2}{dt} = \gamma_p \frac{dy}{dt} = \gamma_p V_y \\ V^3 = \frac{dx^3}{d\tau} = \gamma_p \frac{dx^3}{dt} = \gamma_p \frac{dz}{dt} = \gamma_p V_z \end{cases} \Rightarrow \overline{U} = \gamma_p(c, \vec{V})$$ (1pts)

$$\begin{aligned}
\overline{U}^2 &= U^\alpha U_\alpha = U^0 U_0 + U^i U_i \quad , \ i = 1,2,3 \\
&= (\gamma_p)^2 c^2 - (\gamma_p)^2 \overrightarrow{(V)}^2 \\
&= c^2 (\gamma_p)^2 (1 - \frac{V^2}{c^2})(\gamma_p)^2 c^2 - (\gamma_p)^2 \overrightarrow{(V)}^2 \\
&= c^2 (\gamma_p)^2 (1 - \frac{V^2}{c^2}) \\
\overline{U}^2 &= V^\alpha V_\alpha = c^2 \longrightarrow \text{un invariant.}
\end{aligned}$$

(2 x 0,5pts)

(f) L'expression du quadrivecteur accélération :

$$\begin{aligned}
\bar{a} &= \frac{d\overline{U}}{d\tau} \Rightarrow a^\alpha = \frac{dV^\alpha}{d\tau} = \frac{d^2 x^\alpha}{d\tau^2} \\
\bar{a}\overline{U} &= a^\alpha U_\alpha = \frac{dV^\alpha}{d\tau} V_\alpha = \frac{1}{2}\frac{d}{d\tau}(V^\alpha V_\alpha) = \frac{1}{\tau}\frac{d}{d\tau}(c^2) = 0
\end{aligned}$$

(2 x 0,5pts)

$\bar{a}\overline{U}$ sont toujours orthogonaux. (0,5pts)

2. **Collision élastique, Effet Compton:**

(a) $\gamma \ + \ e^- \longrightarrow \ \gamma \ + \ e^-$ $(0, 5pts)$

P1 P2 P3 P4

$$(P)_\gamma + (P)_{e^-} = (P')_\gamma + (P')_{e^-} \Rightarrow (P)_\gamma + (P)_{e^-} - (P')_\gamma = (P')_{e^-} \qquad (1)$$

Avant collision :

$$(P)_\gamma = \begin{pmatrix} \frac{E}{c} \\ \frac{E}{c}\overrightarrow{u} \end{pmatrix}, (P)_{e^-} = \begin{pmatrix} m_{e^-} c \\ \overrightarrow{0} \end{pmatrix}, (P')_\gamma = \begin{pmatrix} \frac{E'}{c'} \\ \frac{E'}{c'}\overrightarrow{u'} \end{pmatrix}, (P')_{e^-} = \begin{pmatrix} \gamma_{e^-} m_{e^-} c \\ \overrightarrow{P'} \end{pmatrix}$$

(4 x 0,25pts)

$$\begin{aligned}
(1)^2 &\Rightarrow \left[P_\gamma + P_{e^-} - P'_\gamma\right]^2 = (P'_{e^-})^2 \\
&\Rightarrow P_\gamma^2 + P_{e^-}^2 + (P'_\gamma)^2 + 2P_\gamma P_{e^-} - 2P_\gamma P'_\gamma - 2P_{e^-} P'_\gamma = (P'_{e^-})^2
\end{aligned}$$

$$(P_\gamma)^2 = 0 \ , \ (P'_\gamma)^2 = 0 \ , \ (P_{e^-})^2 = m_{e^-}^2 c^2 \ , \ (P'_{e^-})^2 = m_{e^-}^2 c^2$$ (4 x 0,25pts)

$$P_\gamma P_{e^-} - P_\gamma P'_\gamma - P_{e^-} P'_\gamma = 0 \qquad (2)$$

$$
\begin{aligned}
P_\gamma P_{e^-} &= \begin{pmatrix} \frac{E}{c} \\ \frac{E}{c}\overrightarrow{u} \end{pmatrix} \begin{pmatrix} m_{e^-}c \\ \overrightarrow{0} \end{pmatrix} = m_{e^-}E, \\
P_\gamma P'_\gamma &= \begin{pmatrix} \frac{E'}{c'} \\ \frac{E'}{c'}\overrightarrow{u'} \end{pmatrix} \begin{pmatrix} \frac{E}{c} \\ \frac{E}{c}\overrightarrow{u} \end{pmatrix} = \frac{EE'}{c^2} - \frac{EE'}{c^2}\overrightarrow{u}\,.\,\overrightarrow{u'} \\
&= \frac{EE'}{c^2}(1 - cos\ \theta), \\
P_{e^-}P'_\gamma &= \begin{pmatrix} m_{e^-}c \\ \overrightarrow{0} \end{pmatrix} \begin{pmatrix} \frac{E'}{c'} \\ \frac{E'}{c'}\overrightarrow{u'} \end{pmatrix}
\end{aligned}
$$

(4 x 0,25pts)

(b) En remplaçant dans (2) :

$$
\begin{aligned}
m_e(E - E') = \frac{EE'}{c^2}(1 - cos\ \theta) &\Rightarrow (\frac{1}{E'} - \frac{1}{E}) = \frac{1}{m_{e^-}c^2}(1 - cos\ \theta) \\
E = h\nu = \frac{hc}{\lambda} &\Rightarrow \lambda' - \lambda = \frac{h}{m_{e^-}c^2}(1 - cos\ \theta)
\end{aligned}
$$

(2 x 0,25pts)

3. **Formalisme tensoriel en relativité:**

(a) Le carré d'un quadri-vecteur en formalisme tensoriel:

$$\overline{\omega}^2 = \omega^\alpha \omega_\alpha = \omega^0\omega_0 - (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2$$

(1pts)

(b)

$$\begin{cases} (x')^1 = x^1 \cos\theta + x^2 \sin\theta \\ (x')^2 = -x^1 \sin\theta + x^2 \cos\theta \end{cases}$$

$T^{ij} = \begin{pmatrix} (x^2)^2 & -x^1x^2 \\ -x^1x^2 & (x^1)^2 \end{pmatrix}$ est un tenseur d'ordre 2.

$T^{ij} \to T'^{\,ij} = \frac{\partial (x')^i}{\partial x^k}\frac{\partial (x')^j}{\partial x^l}T^{kl}$

$$
\begin{aligned}
T'^{\,11} &= \frac{\partial x'^{\,1}}{\partial x^1}\frac{\partial x'^{\,1}}{\partial x^1}T^{11} + \frac{\partial x'^{\,1}}{\partial x^1}\frac{\partial x'^{\,1}}{\partial x^2}T^{12} + \frac{\partial x'^{\,1}}{\partial x^2}\frac{\partial x'^{\,1}}{\partial x^1}T^{21} + \frac{\partial x'^{\,1}}{\partial x^2}\frac{\partial x'^{\,1}}{\partial x^2}T^{22} \\
&= cos\ \theta.cos\ \theta\ (x^2)^2 + (-sin\ \theta) + (cos\ \theta)(-x^1x^2) + (cos\ \theta)(cos\ \theta)(x^1)^2 = (x^2)^2 \\
&= (-x^1 cos\ \theta + x^2 cos\ \theta)^2 \\
&= cos^2\ \theta\ (x^2)^2 - 2sin\ \theta\ cos\ \theta\ x^1x^2 + sin^2\ \theta\ (x^1)^2
\end{aligned}
$$

(3 x 0,5pts)

Ainsi :

$$T'^{\,11} = \frac{\partial x'^{\,1}}{\partial x^1}\frac{\partial x'^{\,1}}{\partial x^1}T^{11} + \frac{\partial x'^{\,1}}{\partial x^1}\frac{\partial x'^{\,1}}{\partial x^2}T^{12} + \frac{\partial x'^{\,1}}{\partial x^2}\frac{\partial x'^{\,1}}{\partial x^1}T^{21} + \frac{\partial x'^{\,1}}{\partial x^2}\frac{\partial x'^{\,1}}{\partial x^2}\ T^{22}$$

Les autres relations s'obtiennent de la même façon.