L'arbi Ben M'hidi University

Faculty: Exact Sciences, Natural and Life sciences **In:** 17/01/2024
Department : Mathematics and Computer Science. **Duration:** 1h30min
Level: Second year of Bachelor Mathematics

Algebra 3 test

Exercise 1: (6 pts)

1/ Calculate the matrix T^n*, such that* $T = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

2/ Find the matrix $\exp N$*, where* $N = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Exercise 2: (8 pts)

Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$.

1/ Determine the characteristic polynomial $C_A(x)$.
2/ What can you say about the matrix A*? Explain your answer.*
3/ Find the eigenspace corresponding to each eigenvalue.
4/ Can you determine $\ln A$*? Justify your answer.*
5/ Solve the differential system $X' = AX$.

Exercise 3: (6 pts)

1/ In parts (a) and (b) determine whether the statement is true or false, and justify your answer.

(a) *If* A *is a non diagonalizable matrix in* $\mathbb{R}[x]$*, then* A *is trigonalizable in* $\mathbb{R}[x]$.
(b) *If every eigenvalue of a matrix* A *has multiplicity 1, then* A *is diagonalizable.*

2/ In parts (c) and (d) prove the propositions

(c) *If* A, B *and* C *are matrices for which* A *is similar to* B *and* B *is similar to* C*, then* A *is similar to* C.
(d) *If* λ *is an eigenvalue of a square matrix* A*, then* λ^5 *is an eigenvalue of* A^5.

Solution:

Exercise 1:

$$1/\ T^n = \Bigg[\underbrace{\begin{pmatrix}3&0&0\\0&2&0\\0&0&2\end{pmatrix}}_{D} + \underbrace{\begin{pmatrix}0&0&4\\0&0&1\\0&0&0\end{pmatrix}}_{N}\Bigg]^n, \text{ on a } N^2 = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \text{ et donc } N \text{ est nilpotente}$$

d'indice 2.......$(1pt)$

Alors $T^n = \sum_{k=0}^{n} C_n^k D^{n-k} N^k = D^n + nD^{n-1}N.....(1pt)$ so $T^n = \begin{pmatrix}5^n & 0 & 4n5^{n-1}\\ 0 & 2^n & n2^{n-1}\\ 0 & 0 & 2^n\end{pmatrix}....(1pt)$

tel que $nD^{n-1}N = \begin{pmatrix}n5^{n-1} & 0 & 0\\ 0 & n2^{n-1} & 0\\ 0 & 0 & n2^{n-1}\end{pmatrix}\begin{pmatrix}0&0&4\\0&0&1\\0&0&0\end{pmatrix} = \begin{pmatrix}0 & 0 & 4n5^{n-1}\\ 0 & 0 & n2^{n-1}\\ 0 & 0 & 0\end{pmatrix} ...(0.5)$ et $D^n =$

$\begin{pmatrix}5^n & 0 & 0\\ 0 & 2^n & 0\\ 0 & 0 & 2^n\end{pmatrix}.....(0.5)$

2/ $\exp N = I + N + \frac{1}{2!}N^2 + .. + \frac{1}{n!}N^n + ... = I + N...(1pt)$

$$\exp N = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \begin{pmatrix}0&0&4\\0&0&1\\0&0&0\end{pmatrix} = \begin{pmatrix}1&0&4\\0&1&1\\0&0&1\end{pmatrix}...(1pt)$$

<u>Exercise 2:</u> (8 pts)

1/ $C_A(x) = \det(M - xI_3) = -(1-x)(1+x)(3-x)....(2pts)$.

2/ We can say that A is diagonalizable because $C_A(x)$ is split and has simple roots, so $m_A(x) = -C_A(x)$ is also split at simple roots

3/ Find the eigenspaces

$$\text{we obtain} \begin{cases} \text{for } \lambda_1 = 1, E_{\lambda_1} = Vect\{(-1,0,1)\}....(0.75) \\ \text{for } \lambda_2 = -1, E_{\lambda_2} = Vect\{(1,-2,1)\}....(0.75) \\ \text{for } \lambda_3 = 3, E_{\lambda_3} = Vect\{(1,2,1)\}.......(0.75) \end{cases}.$$

4/ We can't determine the matrix $\ln A$, because the function logarithme is not defined on $\lambda_2 = -1$.

5/ We have $X(t) = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + c_3 e^{\lambda_3 t} X_3,\ /\ c_1, c_2, c_3 \in \mathbb{R}...(1pt)$ then the solution of $X' = AX$ is $X(t) = c_1 e^t \begin{pmatrix}-1\\0\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\-2\\1\end{pmatrix} + c_3 e^{3t}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}-c_1e^t + c_2e^{-t} + c_3e^{3t}\\ -2c_2e^{-t} + 2c_3e^{3t}\\ c_1e^t + c_2e^{-t} + c_3e^{3t}\end{pmatrix}$ / $c_1, c_2, c_3 \in \mathbb{R}$.....(1 pt)

<u>Exercise 3:</u> (6 pts)

1/

(a) is false ...(0.25)because we can find an irreductible characteristic polynomial in $\mathbb{R}[x]$, as example for the matrix $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $C(x) = x^2 + 1$, so we can find a non diagonalizable and non trigonalizable matrix in the same time in $\mathbb{R}[x]...(1pt)$

(b) is true ..(0.25)If every eigenvalue of a matrix A has multiplicity 1, then $m_A(x) = -C_A(x)$ will be split at simple roots so A will be diagonalizable....$(1pt)$
2/
(c) A is similar to B so it exist an invertible matrix P such that $A = P.B.P^{-1}$...... $*(0.5)$ and B is similar to C, then it exist an other invertible matrix P' such that $B = P'.C.P'^{-1}$.....$*$ $*(0.5)$
remplace ** in * we obtain $A = P.P'.C.P'^{-1}.P^{-1} = (P.P').C.(P.P')^{-1} / P.P'$ is invertible because P and $P'$are and $(P.P')^{-1} = P'^{-1}.P^{-1}... (1pt)$
So A is similar to C.
(d) λ is an eigen value for a matrix $A \in M_n(\mathbb{k})$ means that exist a non vector V for $\mathbb{k}^n$ such as $A.V = \lambda.V$..... $*(0.5)$
So $A^5.V = A^5(\lambda.V) = A^4\lambda(A.V)$ by * we obtain

$$A^5.V = A^5(\lambda.V) = A^4\lambda(A.V) = A^4\lambda(\lambda.V) = A^4\lambda^2.V$$

by the same methode we obtain

$$\begin{aligned} A^5.V &= A^3\lambda^2(A.V) = A^3\lambda^2(\lambda.V) = A^3.\lambda^3.V \\ &= A^2.\lambda^3(A.V) = A^2\lambda^2(\lambda.V) = A^2.\lambda^3.V \\ &= A\lambda^3(A.V) = A.\lambda^3(\lambda.V) = A\lambda^4V = \lambda^4(A.V) \\ &= \lambda^4(\lambda.V) = \lambda^5V \end{aligned}$$

so λ^5 is an eigen value for $A^5... (1pt)$.