Exercice 2 **(03.5 points).**

1. On a

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\theta^n C_n^k p(1-p)^n = 1, \qquad \mathbf{(0.5)}$$

où

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n} C_n^k p\left[\theta(1-p)\right]^n &= \sum_{n=0}^{\infty} p\left[\theta(1-p)\right]^n \sum_{k=0}^{n} C_n^k \\ &= \sum_{n=0}^{\infty} p\left[\theta(1-p)\right]^n 2^n \qquad \mathbf{(0.25)} \\ &= \sum_{n=0}^{\infty} p\left[2\theta(1-p)\right]^n \\ &= \frac{p}{1-2\theta(1-p)} = 1. \qquad \mathbf{(0.25)}\end{aligned}$$

Donc

$$\theta = \frac{1}{2}. \qquad \mathbf{(0.25)}$$

2. On a

$$\begin{aligned}P(X=k) &= \sum_{n=k}^{\infty} P(X=k, Y=n) \qquad \mathbf{(0.25)} \\ &= \sum_{n=k}^{\infty} C_n^k p\left(\frac{1-p}{2}\right)^n \\ &= \frac{2p}{p+1}\left(\frac{1-p}{1+p}\right)^k. \qquad \mathbf{(0.25)}\end{aligned}$$

Donc

$$\begin{aligned}\varphi_X(t) &= \sum_{k=0}^{\infty} e^{itk} P(X=k) \qquad \mathbf{(0.25)} \\ &= \frac{2p}{p+1}\sum_{k=0}^{\infty} e^{itk}\left(\frac{1-p}{1+p}\right)^k. \\ &= \frac{2p}{p+1}\sum_{k=0}^{\infty}\left[\frac{1-p}{1+p}e^{it}\right]^k \\ &= \left(\frac{2p}{p+1}\right)\frac{1}{1-\frac{1-p}{1+p}e^{it}} \qquad \mathbf{(0.25)} \\ &= \frac{2p}{1+p+(p-1)e^{it}}. \qquad \mathbf{(0.25)}\end{aligned}$$

On a

$$E(X) = \frac{\varphi_X'(0)}{i}, \qquad \mathbf{(0.25)}$$

où

$$\varphi_X'(t) = \frac{-2i(p-1)pe^{it}}{\left(1+p+(p-1)e^{it}\right)^2} \qquad \mathbf{(0.25)} \Longrightarrow \varphi_X'(0) = \frac{-i(p-1)}{2p}, \qquad \mathbf{(0.25)}$$

alors

$$E(X) = \frac{1-p}{2p}. \qquad \mathbf{(0.25)}$$