alors

$$\begin{aligned} g_Z(s) &= \int_{-3}^{-9/5} \frac{5}{6} e^{sz} dz \\ &= \frac{5}{6s}\left(e^{-\frac{9}{5}s} - e^{-3s}\right), \quad \color{red}{\mathbf{(0.5)}} \end{aligned}$$

et

$$g_Z'(s) = \frac{-5}{6s^2}\left(e^{-\frac{9}{5}s} - e^{-3s}\right) + \frac{5}{6s}\left(3e^{-3s} - \frac{9}{5}e^{-\frac{9}{5}s}\right). \quad \color{red}{\mathbf{(0.5)}}$$

En utilisant la règle de l'Hôpital, on obtient

$$\begin{aligned} g_Z'(s)\big|_{s=0} &= \frac{-5}{12}\left(\frac{81}{25}e^{-\frac{9}{5}s} - 9e^{-3s}\right)\bigg|_{s=0} + \frac{5}{6}\left(\frac{81}{25}e^{-\frac{9}{5}s} - 9e^{-3s}\right)\bigg|_{s=0} \\ &= -\frac{12}{5}. \quad \color{red}{\mathbf{(0.25)}} \end{aligned}$$

Donc

$$E(Z) = -\frac{12}{5}. \quad \color{red}{\mathbf{(0.25)}}$$

5. On a

$$f_{UV}(u,v) = |J|\, f_{XY}\left(h^{-1}(u,v), g^{-1}(u,v)\right). \quad \color{red}{\mathbf{(0.5)}}$$

Faisons le changement de variables $(u,v) = \left(h(x,y) = \frac{x}{y},\ g(x,y) = x+y\right)$. La réciproque est

$$(x,y) = \left(h^{-1}(u,v) = \frac{uv}{1+u}, g^{-1}(u,v) = \frac{v}{1+u}\right). \quad \color{red}{\mathbf{(0.5)}}$$

Le Jacobien J est

$$J = \det\begin{pmatrix} \frac{v}{(1+u)^2} & \frac{u}{1+u} \\ \frac{-v}{(1+u)^2} & \frac{1}{1+u} \end{pmatrix} = \frac{v}{(1+u)^2}. \quad \color{red}{\mathbf{(0.5)}}$$

Donc

$$\begin{aligned} f_{UV}(u,v) &= \frac{v}{(1+u)^2}\frac{5}{18}\frac{1}{\sqrt{\frac{uv}{1+u}}\sqrt{\frac{v}{1+u}}} \\ &= \frac{5}{18(u+1)\sqrt{u}}. \quad (u,v) \in D. \quad \color{red}{\mathbf{(0.5)}} \end{aligned}$$