Oum El Bouaghi University
Department of mathematics and informatics
L3 Mathematics
Correction -Optimization-

Exercise 01 *(06pts)*
Let $f(x)$ be a differentiable function on a convex set $C \subset \mathbb{R}^n$.
1- By the definition of convex function, we have

$$\begin{aligned} f(x+t(y-x)) &= f(ty+(1-t)x)tf(y)+(1-t)f(x), \forall x,y \in C, \forall t \in (0,1), \\ & So \\ & f(x+t(y-x))-f(x) \leq t(f(y)-f(x)). \end{aligned}$$

Devide on t and the limit to 0

$$\langle \nabla f(x), y-x \rangle = \lim_{t \to 0} \frac{f(x+t(y-x))-f(x)}{t} = f(y)-f(x)$$

Then,

$$f(y) \geq f(x) + \langle \nabla f(x), y-x \rangle, \forall x,y \in C$$

-

$$f(x) \ convex \Rightarrow f(y) \geq f(x) + \langle \nabla f(x) \quad (y-x) \rangle, \ \ \forall x,y \in C$$

2- By (1) $\begin{cases} f(y) \geq f(x) + \langle \nabla f(x), y-x \rangle, \\ f(x) \geq f(y) + \langle \nabla f(y), x-y \rangle, \end{cases}$
We collect the both inequalities, we find the result

$$\langle \nabla f(y) - \nabla f(x) \quad y-x \rangle \geq 0, \ \ \forall x,y \in C$$

Exercise 02 *(07pts)*

$$f(x,y) = e^{x-y}(x^2-2y^2)$$

- The Gradient is $\nabla f(x,y) = \begin{pmatrix} 2xe^{x-y} - e^{x-y}\left(2y^2-x^2\right) \\ e^{x-y}\left(2y^2-x^2\right) - 4ye^{x-y} \end{pmatrix}$
- The critical points $(\nabla f(x,y)=0)$ That's mean $(x,y)=(0,0)$ and $(x,y)=(-4,-2)$, we have two points $M_1=(0,0)$ and $M_2=(-4,-2)$
- The nature (Calculate the Hessian)

Hessian is $\begin{pmatrix} 2e^{x-y} - e^{x-y}\left(2y^2-x^2\right) + 4xe^{x-y} & e^{x-y}\left(2y^2-x^2\right) - 2xe^{x-y} - 4ye^{x-y} \\ e^{x-y}\left(2y^2-x^2\right) - 2xe^{x-y} - 4ye^{x-y} & 8ye^{x-y} - e^{x-y}\left(2y^2-x^2\right) - 4e^{x-y} \end{pmatrix}$

For M_1 $\quad H_f(M_1) = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$, $a_{11}=2$, and $det(H)=-8 \Rightarrow M_1$ is col point.

For M_2 $\quad H_f(M_2) = \begin{pmatrix} -6e^{-2} & 8e^{-2} \\ 8e^{-2} & -12e^{-2} \end{pmatrix}$, $a_{11} = -6e^{-2} < 0$, and $det(H) = 8e^{-4} > 0 \Rightarrow M_1$ is a maximum point.

Exercise 03 *(07pts)*
Consider the function

$$f(x, y) = x^2 + 2y^2 - 2xy + 2x + 1$$

Using Gradient method with fixed step
(Put $\quad X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \rho = 0.1, \quad \varepsilon = 10^{-7}$)
The partial derivatives are: $\begin{cases} f_x = 2x - 2y + 2 \\ f_y = 4y - 2x \end{cases}$
The iterations are as follows:

$$\begin{aligned} (x_1, y_1) &= (x_0, y_0) - \rho\nabla f(x_0, y_0) = (0.8, 0.8) \\ Errors1 &= 0.14 \text{ GO TO STEP 2} \\ (x_2, y_2) &= (x_1, y_1) - \rho\nabla f(x_1, y_1) = (0.6, 0.64) \\ Errors1 &= 0.14 \text{ GO TO STEP 3} \\ (x_3, y_3) &= (x_2, y_2) - \rho\nabla f(x_2, y_2) = (0.408, 0.504) \end{aligned}$$