Larbi Ben Mhidi University - Oum El Bouaghi-

Faculty of exact sciences, natural and life sciences

Department of Mathematics and Computer Science **Academic year: 2023/2024**

Level: 1st year MI **Duration:1h30**

Exam: Analysis 1

Exercise 1 (points)

1) Let a and b be two numbers where $a \in \mathbb{Q}^+$, $b \in \mathbb{Q}^+$and $\sqrt{ab} \notin \mathbb{Q}^+$. Prove that $\sqrt{a} + \sqrt{b} \notin \mathbb{Q}^+$.

2) Let A be a subset of $\mathbb{R}$ bounded from above, we know the set $-A = \{-x; x \in A\}$. Prove that: $\inf(-A) = -\sup A$.

3) Prove that: $\forall x \in \mathbb{R} :$ arc tan x + arc cotan $x = \frac{\pi}{2}$.

4) Write the expression $L(x) = \sin^3 x \cos^3 x$ in linear form (Note that: $\sin x \cos x = \frac{1}{2} \sin 2x$).

Exercise 2 (points)

Let f be a function defined in the interval $I = [2, +\infty[$ by $f(x) = \ln x - \frac{1}{x} + 2$.

1) a) Prove that the function f is strictly increasing on I.

b) Using the Lagrange's finite-increment theorem Prove that: $\forall a, b \in I: |f(b) - f(a)| \leq \frac{3}{4}|b - a|$.

2) Let $(v_n)_{n\in\mathbb{N}}, (u_n)_{n\in\mathbb{N}}$ be a sequences defined by: $\forall n \in \mathbb{N}: \begin{cases} u_0 = 2 \\ u_{n+1} = f(u_n) \end{cases}$; $\begin{cases} v_0 = 3 \\ u_{n+1} = f(v_n) \end{cases}$

a) Study the monotonicity of the two sequences $(v_n)_{n\in\mathbb{N}}, (u_n)_{n\in\mathbb{N}}$

b) Prove that: $\forall n \in \mathbb{N}: |v_{n+1} - u_{n+1}| \leq \frac{3}{4}|v_n - u_n|$.

c) Using the proof by induction, prove that: $\forall n \in \mathbb{N}: |v_n - u_n| \leq \left(\frac{3}{4}\right)^n$.

3) Prove that $(v_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ are adjacent.

4) We put $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \ell$.

a) Prove that $\forall n \in \mathbb{N}: |\ell - u_n| \leq |v_n - u_n|$. b) Deduce a value rounded to 10^{-2} for ℓ.

Exercise 3 (points) Let f be a function defined in the interval $\mathbb{R}$by $g(x) = \begin{cases} x^2 \sin\frac{1}{x}, x > 0 \\ e^{x^2} - \cos x, x \leq 0 \end{cases}$.

1) Examine the continuity of g over $\mathbb{R}$. 2) Using L'Hopital's rule, calculate $\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x}$.

3) Examine the derivability of g over $\mathbb{R}$. 4) Express $g'(x)$ in terms of x.

5) Is the function g of class C^1 on $\mathbb{R}$? justify your answer.

6) Using L'Hopital's rule, calculate $\lim_{x\to+\infty} [g(x) - x]$, What do you conclude?

Good luck.