1) * T linéaire ? $\forall f, g \in E$, $\forall \alpha, \beta \in \mathbb{K}$.

$$\begin{cases} a)\ \big(T(f+g)\big)(t) \overset{?}{=} (Tf)(t) + (Tg)(t) \\ b)\ T(\alpha f)(t) \overset{?}{=} \alpha T(f)(t). \end{cases}$$

a) $T(f+g)(t) = \cos(t) \int_0^t \sin(s)(f+g)(s)\,ds =$

$\cos(t) \int_0^t \sin(s) f(s)\,ds + \cos(t) \int_0^t \sin(t) g(s)\,ds =$

$(Tf)(t) + (Tg)(t)$.

1

b) $T(\alpha f)(t) = \cos(t) \int_0^t \sin(s)(\alpha f)(s)\,ds =$

$\alpha \cos(t) \int_0^t \sin(s) f(s)\,ds = \alpha (Tf)(t)$. $\Rightarrow$ T linéaire.

*) T borné ? $\forall f \in E$; $\exists k > 0$; $\|Tf\| \leq k \|f\|$

2

on a ; $|(Tf)(t)| = |\cos(t)| \left| \int_0^t \sin(s) f(s)\,ds \right| \leq \int_0^t |\sin(s)| |f(s)|\,ds$

$\leq \int_0^{\pi/2} |f(s)|\,ds$ car $|\cos(t)| \leq 1$ et $|\sin(s)| \leq 1$

en applique inégalité de Cauchy-Shwarz on trouve :

$$\int_0^{\pi/2} |f(s)|\,ds \leq \left(\int_0^{\pi/2} 1^2\,ds \right)^{1/2} \left(\int_0^{\pi/2} |f(s)|^2\,ds \right)^{1/2} = \sqrt{\frac{\pi}{2}} \|f\|_2$$

$\Rightarrow \|Tf\| \leq \frac{\pi}{2} \|f\|$ $\Rightarrow$ T borné et $\|T\| \leq \frac{\pi}{2}$

(2)